The topological phase factor in Van-Vleck's formula:

Morse thm:

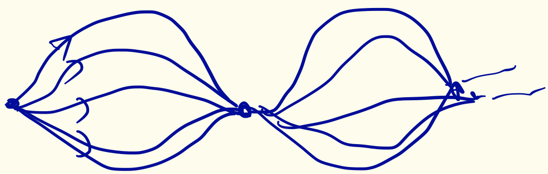
ν = # of negative eigenvalues of $\frac{\delta^2 S}{\partial x_i \partial x_j}$

= # of conjugate points along trajectory from x_N to x_0.

Conjugate points:

$$\boxed{\frac{\partial x(p_0, t)}{\partial p_0} = 0}$$

Different trajectories coincide at the conjugate points.



$$-\frac{\partial^2 S}{\partial x_0 \partial x_N} = \mathcal{N}^{-1}(p_0, t) = \infty$$

at conjugate points.

For SHO: $\mathcal{N} = \frac{\sin(\omega T)}{m\omega}$

$$\omega T = n\pi$$

$\Rightarrow$ SHO propagator:

$$= e^{-i\pi/4} \, e^{-\frac{i\pi}{2}\left[\frac{\omega T}{\pi}\right]} \sqrt{\frac{m\omega}{2\pi|\sin\omega T|}}$$

$$e^{\frac{i m\omega}{2\sin(\omega t)}\left\{(x_N^2 + x_0^2)\cos(\omega t) - 2x_N x_0\right]}$$

More details on conjugate points:

$$\delta x(p_0, t) = -\left[\frac{\partial^2 S}{\partial x\, \partial x_0}\right]^{-1} \delta p_0$$

$$= 0 \quad \text{at conjugate points.}$$

SHO conjugate points

$$x_{cl}(t) = \frac{1}{\sin(\omega T)}\left[x_N \sin(\omega t) + x_0 \sin\omega(T-t)\right]$$

$$x_{cl}(t=0) = x_0$$

$$x_{cl}(t=T) = x_N$$

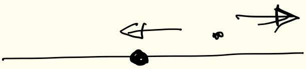
Consider trajectories

$$x(p_0, t) = \frac{x_0}{\sin(\omega T)}\left\{(-)^n \sin(\omega t) + \sin(\omega T - \omega t)\right]$$

$$x^{even}_{odd}(p, 0) = x_0$$

$$\dot{x}^{0}_{even/odd} = \frac{x_0\omega}{\sin\omega T}\left\{\pm 1 - \cos(\omega T)\right]$$

$$x_{even}\left(p_0, \frac{n\pi}{\omega}\right) = x_{odd}\left(p_0, \frac{n\pi}{\omega}\right)$$
$$= (-)^n x_0$$



$$x(p_0, t) = \frac{x_0}{\sin(\omega T)} \{ (-)^n \sin(\omega t) + \sin(\omega T - \omega t)]$$

$$\dot{x} = \frac{x_0}{\sin(\omega T)} [(-)^n \omega \cos(\omega t) - \omega \cos(\omega T - \omega T)]$$

$$\dot{x}(t=0) = \frac{x_0 \omega}{\sin(\omega T)} [(-)^n - \cos(\omega T)]$$

$$= \frac{x_0 \omega}{\sin(\omega T)} [\pm 1 - \cos(\omega T)]$$

$$x\left(p_0, t = \frac{n\pi}{\omega}\right) = \frac{x_0}{\sin(\omega T)} \{ \sin(\omega T - n\pi)]$$
$$= (-)^n x_0$$

Example # 1: Free Particle

$$K = \langle x_N | e^{-\frac{ip^2 t}{2m}} | x_0 \rangle$$

$$= \int dp \; e^{ip\cdot(x_N - x_0) + \frac{ip^2 t}{2m}}$$

$$= \sqrt{\frac{m}{2i\pi t}} \; e^{\frac{im}{2}\frac{(x_N - x_0)^2}{t}}$$

Example # 2: Harmonic Oscillator

$$K(x_N, x_0 ; t) = \int D x(t) \; e^{\frac{i}{\hbar}\int_0^t m(\dot{x}^2 - m^2 w^2)\, dt'}$$

$$= F(T) \; e^{i S_{classical}}$$

$$S_{classical} = \frac{1}{2} m \int_0^t [\dot{x}_{cl}^2 - w^2 x_{cl}^2]\, dt'$$

where $x_{cl}(t') = \frac{1}{\sin(wt)} [x_N \sin(wt') + x_0 \sin w(t - t')]$

Evaluating this :

$$S_{cl} = \frac{m\omega}{2\sin(\omega t)} \left[(x_N^2 + x_0^2)\cos(\omega t) - 2 x_N x_0 \right]$$

The prefactor is non-trivial to determine.

$$F(t) = \int Dy(t') \, e^{\frac{i}{\hbar} \int_0^t \frac{1}{2} m [\dot{y}^2 - \omega^2 y^2] dt}$$

$$y(t) = y(0) = 0$$

write $y(t') = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi t'}{t}$

$$\int_0^t \frac{1}{2} m (\dot{y}^2 - \omega^2 y^2) dt'$$

$$= \frac{1}{2} m \frac{t}{2} \sum_n \left[\left(\frac{n\pi}{t}\right)^2 - \omega^2 \right] a_n^2$$

$$\Rightarrow F(t) = \int da_n \exp \frac{imt}{4} \sum_n \left[\left(\frac{n\pi}{t}\right)^2 - \omega^2 \right] a_n^2$$

$$= \text{constant} \prod_{n=1}^{N} \left[1 - \frac{\omega^2 t^2}{n^2 \pi^2} \right]^{-1/2} = \text{const} \left[\frac{\omega t}{\sin \omega t} \right]^{-1/2}$$

$$= \left[\frac{m\omega}{2\pi i \sin \omega t} \right]^{1/2}$$

Using van-Vleck's formula:

$$\frac{\partial^2 S}{\partial x_N \partial x_0} = \frac{m\omega \times -2}{2\sin(\omega t)}$$

$$F = \left[-\frac{1}{2\pi i} \frac{\partial^2 S}{\partial x_N \partial x_0}\right]^{1/2}$$

$$= \left[\frac{m\omega}{2\pi i \sin[\omega t]}\right]^{1/2}$$

Let's revisit the free problem:

$$S_{cl} = \frac{1}{2} m \left[\frac{(x_N - x_0)^2}{t}\right]$$

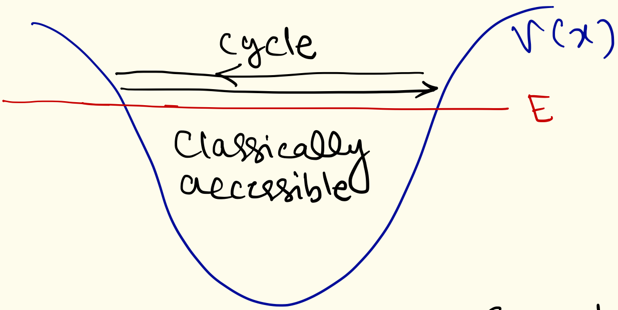
$$\frac{\partial^2 S_{cl}}{\partial x_b \partial x_a} = -\left[\frac{m}{t}\right]$$

$$\Rightarrow F = \left[\frac{1}{2\pi i} \frac{m}{t}\right]^{1/2}$$

WKB Condition from Single Valued-ness of Wavefunction/Propagator

WKB wave-fn:

$$\psi \sim \frac{1}{\sqrt{p(x)}}\, e^{iS(x,E)}$$



Under a full cycle, S changes by $\oint p\,dx$. On the other hand, at each turning point $p \to -p \Rightarrow$ phase $\sim \left[\sqrt{p(x)}\right]^{-1}$ changes by $e^{i\pi/2}$.

For single-valued-ness of ψ

$$\oint p\,dx = 2\pi n + \frac{\pi}{2} \times \text{no. of soft turning points} + \pi \times \text{no. of hard turning points.}$$

$$= 2\pi \left[n + \frac{\mu}{4} + \frac{b}{2} \right]$$

μ = # of soft turning points encountered during a full cycle

b = # hard turning points encountered during a full cycle.

Relation to Van-Vleek formula.

Recall the Van Vleek formula.

$$K(x_N, x_0, t) = F(t)\, e^{i S_{\text{classical}}}$$

$$f(t) = \sqrt{\frac{\partial^2 S}{\partial x_N \partial x_0}} = \sqrt{\left| \frac{\partial^2 S}{\partial x_N \partial x_0} \right|}\; e^{\frac{i\pi}{2} \times \text{no. of conjugate points.}}$$

Consider propagator for a full cycle.

From above discussion,

no. of conjugate points during a full cycle = no. of soft turning points.

from Van Vleck to Gutzwiller (Outline)

$$G(x_N, x_0, E)$$

$$= \int_t K(x_N, x_0, t)\, e^{iEt}$$

$$= e^{-i n_{conj.} \frac{\pi}{2}} \int dt \sqrt{\left| \frac{\partial^2 S}{\partial x_N \partial x_0} \right|} \; e^{iS(x_N, x_0, t) + iEt}$$

Since $\hbar \to 0$, we can again do a saddle point integration.

The saddle point time t_0 is

$$\left.\frac{\partial S}{\partial t}\right|_{t_0} + E = 0$$

But this is just the usual relation between E and S in classical mechanics

$$\tilde{S} = S + E t_0$$

$$\tilde{S} = S + E t_0$$

$$= \int_0^{t_0} \mathcal{L}\, dt + E t_0$$

$$= \int_0^{t_0} [p \dot{q} - H]\, dt + E t_0$$

$$= \int_0^{t_0} p \dot{q}\, dt = \int_0^{t_0} p\, dq$$

Doing the saddle point integration,

$$G(x_N, x_0, E) \sim e^{-\frac{i\pi}{2} n_{\text{conj}} + i\tilde{S}(E, x_N, x_0)}$$

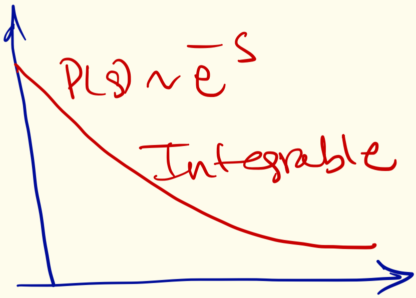
$$\sqrt{\left|\frac{\partial^2 S}{\partial x_N \partial x_0}\right|} \Big/ \sqrt{\frac{\partial^2 S}{\partial T^2}}$$

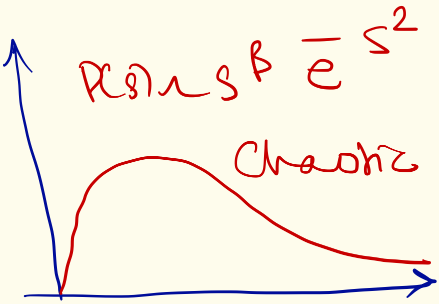
Taking the trace over G picks up $e^{i\tilde{S}}$ over closed orbits.

Quantum Chaos and Level Statistics

Gutzwiller trace formula relates periodic orbits to the distribution of eigenvalues.

Consider the distribution of nearest-neighbour level spacing $P(s)$. (s = spacing)





Riemann-zeta fn and Gutzwiller trace formula

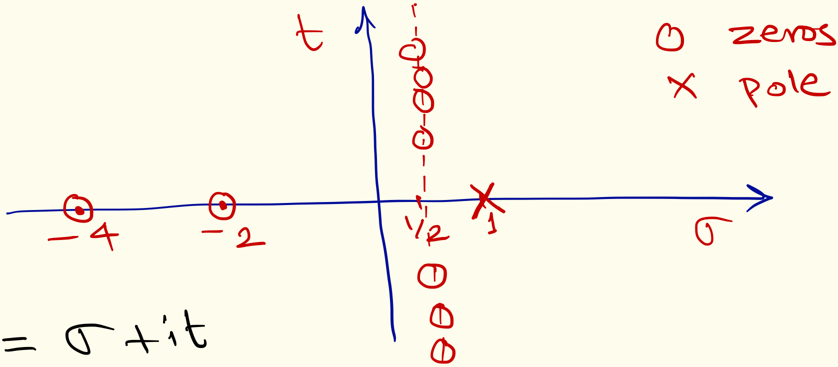
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Since $\quad n = \prod_p p^{m_p(n)}$

$$\Rightarrow \quad \zeta(s) = \sum_{m_p} \frac{1}{\left[\prod_p p^{m_p} \quad s\right]}$$

$$= \sum_{m_p} \prod_p \frac{1}{p^{m_p s}}$$

$$= \prod_p \left[\frac{1}{\left[1 - \frac{1}{p^s}\right]}\right]$$



$$s = \sigma + it$$

Riemann conjecture: All zeros with negative imaginary component have

$$\sigma = 1/2$$

Zeros at $\sigma = 1/2$ behave like eigenvalues of a chaotic Hamiltonian.

$$\log \zeta(s) = -\sum_p \log\left[1 - \frac{1}{p^s}\right]$$

$$= +\sum_p \sum_k \frac{p^{-sk}}{k}$$

$$= -\sum_p \sum_k \frac{p^{-k(\sigma+it)}}{k}$$

$$= -\sum_p \sum_{k \in \mathbb{Z}} \frac{\exp[-ikt \log p]}{k\, p^{k\sigma}}$$

$$\frac{1}{\pi} \frac{d \operatorname{Im}\left[\log \zeta(s)\right]}{dt} = -\frac{1}{\pi} \sum_{p,k} \frac{\cos(kt \log p) \log(p)}{p^{k\sigma}}$$

Compare this with Gutzwiller formula.

$$\delta g_{osc} = \sum_k \cos\left(k \times \text{action of periodic orbit}\right) \times \text{period}$$

One orbit for each prime p

action $= [t \log p] \sim$ Eisen Energy $\times$ time period

period $T = \log p$

$$\# \text{ of primes } (p < x) \sim \frac{x}{\log x}$$

$\Rightarrow$ No. of period orbits with period $T < T_p \sim \dfrac{e^T}{\log(e^T)}$

$$\sim e^T / T$$

<u>General property</u> of chaotic systems.

$\#$ of periodic orbits of Period T

$\sim e^{sT}$ $s =$ "Topological entropy"

$$\zeta(s) \sim \frac{\xi(s)}{s-1} \leftarrow \text{analytic}$$

$$g(t) = \sum_n \delta(t - t_n)$$

location of zeros on the imag axis

$$\xi(s) = |\xi(s)| e^{i\phi_\sigma(t)} = \xi(0) \prod_n \left[1 - \frac{s}{s_n}\right]$$

$$\frac{d\phi_\sigma}{dt} = \frac{d}{dt} \operatorname{Im}\left[\log \xi(s)\right] \qquad s_n = \frac{1}{2} + it_n$$

$$= -\sum_n \frac{\left[\sigma - 1/2\right]}{(\sigma - 1/2)^2 + (t - t_n)^2}$$

as $\sigma \to 1/2$

$i(t_n - t)$

$$\longrightarrow \sum_n \delta(t - t_n)$$

$\log re^{i\theta}$

$$\operatorname{Im}\log\left[1 - \frac{\sigma + it}{\frac{1}{2} + it_n}\right] = \log\left\{\frac{\left[\frac{1}{2} - \sigma + i(t_n - t)\right]\left[\frac{1}{2} - it_n\right]}{\frac{1}{4} + t_n^2}\right\}$$