A few illustrations of non-degenerate perturbation theory

Example 1: Consider a spin-1/2 system with $H = -\hat{Z} - h\hat{X}$ with $h \ll 1$. We can solve for the eigenvectors and eigenvalues of H exactly as discussed earlier, and then verify that the perturbative result matches with the exact answer.

Exact solution: The two eigenstates are given by $|E_0'\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$

and $|E_1'\rangle = \begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix}$

where $\tan\theta = h$. The corresponding eigenvalues are $E_0' = -\sqrt{1+h^2}$, $E_1' = +\sqrt{1+h^2}$.

To compare with the perturbative solution, lets expand the exact eigenenergies to $O(h^2)$ and exact eigenstates to $O(h)$

$$E_0' \simeq -\left[1 + \frac{h^2}{2}\right], \quad |E_0'\rangle \simeq \begin{bmatrix} 1 \\ \frac{h}{2} \end{bmatrix}$$

$$E_1' \simeq 1 + \frac{h^2}{2}, \quad |E_1'\rangle \simeq \begin{bmatrix} -\frac{h}{2} \\ 1 \end{bmatrix}.$$

Perturbative solution :

We write $H = H_0 + h H_1$ where

$H_0 = -\hat{Z}$, and $H_1 = -\hat{X}$.

The eigenstates and eigenvalues of the unperturbed Hamiltonian H_0 are

$$|E_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad E_0 = -1 \ ;$$

$$|E_1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \qquad E_1 = +1 .$$

The perturbed eigenenergies to $O(h^2)$ are therefore,

$$E_0' = E_0 + h\langle E_0|\hat{X}|E_0\rangle + h^2 \frac{|\langle E_0|\hat{X}|E_1\rangle|^2}{E_0 - E_1} + O(h^3)$$

Now $\langle E_0 | \hat{x} | E_0 \rangle = 0$

and $\langle E_0 | \hat{x} | E_1 \rangle = 1 \Rightarrow$

$$E'_0 = E_0 - \frac{h^2}{2} + o(h^3) = -1 - \frac{h^2}{2} + o(h^3)$$

while $E'_1 = E_1 + h \langle E_1 | \hat{x} | E_1 \rangle + h^2 \dfrac{|\langle E_1 | \hat{x} | E_0 \rangle|^2}{E_1 - E_0}$

$$= 1 + \frac{h^2}{2} + o(h^3).$$

Similarly, $|E'_0\rangle = |E_0\rangle - \dfrac{h \langle E_1 | \hat{x} | E_0 \rangle}{E_0 - E_1} |E_1\rangle + o(h^2)$

$$= |E_0\rangle + \frac{h}{2} |E_1\rangle$$

$$= \begin{bmatrix} 1 \\ \frac{h}{2} \end{bmatrix} + o(h^2)$$

while $|E'_1\rangle = |E_1\rangle - h\dfrac{\langle E_0|\hat{x}|E_1\rangle\, |E_0\rangle}{E_1 - E_0} + O(h^2)$

$$= |E_1\rangle - \frac{h}{2}|E_0\rangle + O(h^2)$$

$$= \begin{bmatrix} -\frac{h}{2} \\ 1 \end{bmatrix} + O(h^2)$$

Therefore, the perturbative results agree with the exact results to the order we are working at.

Example 2: Consider the forced harmonic oscillator,

$$H = \frac{\hat{x}^2 + \hat{p}^2}{2} - F\hat{x}$$

$$= H_0 - F\hat{x}$$

where $F \ll 1$.

One can again find the exact solution by completing the square:

$$H = \frac{(\hat{x} - F)^2}{2} + \frac{\hat{p}^2}{2} - \frac{F^2}{2}$$

$$= \frac{\hat{x}'^2}{2} + \frac{p^2}{2} - \frac{F^2}{2}$$

where $x' = x - F$. Since $[x', p] = -i$, the spectrum of H' is $E_n = \left(n + \frac{1}{2}\right) - \frac{F^2}{2}$ with $n = 0, 1, 2, \ldots \infty$.

while the corresponding eigenstates are given by

$$|n\rangle = e^{-iF\hat{p}} |n\rangle_0$$ where

$|n\rangle_0$ is the unperturbed eigenstate.

This is because $U^+ H U = H_0 - \frac{F^2}{2}$

where $U = e^{-iF\hat{p}}$ (recall that momentum operator generates translation).

Let's verify these exact results using perturbation theory.

Perturbative solution:

$$E_n = E^0_n + {}_0\langle n | -Fx | n\rangle_0 + \sum_{m \neq n} \frac{|{}_0\langle n | Fx | m\rangle_0|^2}{E^0_n - E^0_m}$$

Now, $x = \frac{a + a^{+}}{\sqrt{2}}$ and therefore

$$\,_0\langle n | x | m \rangle_0 = \frac{\delta_{m,n+1}}{\sqrt{2}} \sqrt{n+1}$$

$$+ \frac{\delta_{m,n-1}}{\sqrt{2}} \sqrt{n}$$

$\Rightarrow$ the term linear in F vanishes.

$$\Rightarrow E_n = E^0_n + \frac{F^2}{2} \left[\frac{n+1}{-1} + \frac{n}{1} \right]$$

$$= E^0_n - \frac{F^2}{2}$$

$\Rightarrow E_n = \left(n + \frac{1}{2}\right) - \frac{F^2}{2}$ in agreement

with the exact result.

One may similarly verify that the perturbed state $|n\rangle$ matches (see Shankar 17.2)

Example 3: Next consider an example that is not exactly solvable.

Consider a 1d harmonic oscillator with relativistic kinetic energy

$$H = \sqrt{m^2 + p^2} - m + \frac{x^2}{2} m$$

Let's calculate the change in the ground state energy to the leading order in $1/m$.

By Taylor expansion,

$$H \simeq \underbrace{\frac{p^2}{2m} + \frac{x^2 m}{2}}_{H_0} - \underbrace{\frac{p^4}{8m^3}}_{H_1}$$

Since $\langle n=0| p^4 |n=0\rangle$ is non-zero, the leading order change in the ground state energy is given by,

$$\Delta E_0 = -\frac{1}{8m^3} \langle n=0| p^4 | n=0\rangle$$

$$= \frac{-1}{8m^3}\left[\frac{1}{m\pi}\right]^{1/2} \int dp\, p^4\, e^{-\frac{p^2}{2m}}$$

$$= -\frac{3\sqrt{2}}{8m}$$