Tensor Operators and Wigner-Eckart theorem

Let's recall the definition of a vector operator: a vector operator transforms like a vector under rotation. Specifically, if $\hat{V}_i$ is a vector operator with three components $(\hat{V}_x, \hat{V}_y, \hat{V}_z)$, then under a rotation implemented by the unitary operator $D(R) = e^{-i\hat{J}\cdot\hat{n}\theta}$,

$$D^{\dagger}(R)\,\hat{V}_i\,D(R) = \sum_j R_{ij}\hat{V}_j$$

where R is the 3×3 matrix that implements the rotation. For example, consider a rotation by an angle θ around the $\hat{z}$-axis. For this case, $D(R) = e^{-iJ_z\theta}$ and R is:

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ so that

the rotated operator has components:

$\hat{V}'_x = \hat{V}_x \cos(\theta) - \hat{V}_y \sin(\theta)$ and

$\hat{V}'_y = \hat{V}_y \cos(\theta) + \hat{V}_x \sin(\theta)$, exactly

like the components of an ordinary 3-component vector subjected to rotation around the z-axis. If one chooses the rotation angle θ to be infinitesimal, then the equation $D^\dagger(R)\hat{V}_i D(R) = \sum_j R_{ij} V_j$

becomes $[\hat{V}_i, \hat{J}_j] = i\varepsilon_{ijk}\hat{V}_k$.

i.e. $[\hat{V}_x, \hat{J}_y] = i\hat{V}_z$
$[\hat{V}_y, \hat{J}_z] = i\hat{V}_x$
$[\hat{V}_z, \hat{J}_x] = i\hat{V}_y$ etc.

Let's write these commutation relations in a slightly different way.

Let's define $\hat{V}_0 = \hat{V}_z$, $\hat{V}_{\pm 1} = \mp \frac{(\hat{V}_x \pm i\hat{V}_y)}{\sqrt{2}}$

One may then rewrite the above commutation relations as,

$$[\hat{J}_z, \hat{V}_m] = m\hat{V}_m$$

$$[\hat{J}_\pm, \hat{V}_m] = \sqrt{j(j+1) - m(m\pm 1)}\ \hat{V}_{m\pm 1}$$

where $j = 1$ and $m = 0, \pm 1$.

Note how closely these equations resemble

$$J_z |j, m\rangle = m |j, m\rangle$$

and $J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\ |j, m\pm 1\rangle$

with $j = 1$. Therefore, a vector operator somehow resembles $|j=1, m\rangle$.

However, we can't equate an operator to a state. A precise correspondence is that $\hat{V}_{m_1}|j_2, m_2\rangle$ for any j_2 and m_2 transforms under rotation in the same way as a state $|j_1 = 1, m_1\rangle \otimes |j_2, m_2\rangle$. This directly follows from the above commutation relations:

$$\hat{J}_z \hat{V}_{m_1}|j_2, m_2\rangle$$

$$= \left([\hat{J}_z, \hat{V}_{m_1}] + \hat{V}_{m_1}\hat{J}_z\right)|j_2, m_2\rangle$$

$$= m_1 \hat{V}_{m_1}|j_2, m_2\rangle + \hat{V}_{m_1} m_2 |j_2, m_2\rangle$$

$$= (m_1 + m_2)\hat{V}_{m_1}|j_2, m_2\rangle$$

Compare this with $\hat{J}_z |j_1 = 1, m_1\rangle \otimes |j_2, m_2\rangle$

$$= (\hat{J}_{1z} + \hat{J}_{2z})|j_1 = 1, m_1\rangle \otimes |j_2, m_2\rangle$$

$$= (m_1 + m_2)|j_1 = 1, m_1\rangle \otimes |j_2, m_2\rangle$$

Similarly,

$$J_\pm \hat{V}_{m_1} | j_2, m_2 \rangle$$

$$= ([J_\pm, V_{m_1}] + V_{m_1} J_\pm) | j_2, m_2 \rangle$$

$$= \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)} \; V_{m_1 \pm 1} | j_2, m_2 \rangle$$

$$+ V_{m_1} \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)} \; | j_2, m_2 \pm 1 \rangle$$

where $j_1 = 1$.

Compare this with,

$$J_\pm | j_1 = 1, m_1 \rangle \otimes | j_2, m_2 \rangle$$

$$= (J_{1\pm} + J_{2\pm}) | j_1 = 1, m_1 \rangle \otimes | j_2, m_2 \rangle$$

$$= \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)} \; | j_1 = 1, m_1 \pm 1 \rangle \otimes | j_2, m_2 \rangle$$

$$+ \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)} \; | j_1 = 1, m_1 \rangle \otimes | j_2, m_2 \pm 1 \rangle.$$

Therefore, we conclude that $V_{m_1} | j_2, m_2 \rangle$ indeed transforms like $| j_1 = 1, m_1 \rangle \otimes | j_2, m_2 \rangle$.

$\Rightarrow$ Acting with $\hat{V}_m$ on $|j_2, m_2\rangle$ is akin to <u>adding</u> angular momentum of unit 1 to the state $|j_2, m_2\rangle$. We will soon see consequences of this statement below in Wigner-Eckart theorem.

But first, let us generalize above discussion by introducing tensor operators.

We define a tensor operator of rank j as a set of operators $\hat{O}_{j,m}$ with $m = -j, -j+1, \ldots j-1, j$ (i.e. $2j+1$ components) that satisfy,

$$[\hat{J}_z, \hat{O}_{j,m}] = m\, \hat{O}_{j,m}$$

$$[\hat{J}_\pm, \hat{O}_{j,m}] = \sqrt{j(j+1) - m(m\pm 1)}\; \hat{O}_{j,m\pm 1}.$$

Therefore, a vector operator is just a special case of a tensor operator with $j=1$.

Following a derivation identical to above, a state $\hat{O}_{j_1, m_1} |j_2, m_2\rangle$ transforms under rotation in the same way as a state $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$.

$\Rightarrow$ Acting with $\hat{O}_{j_1, m_1}$ on $|j_2, m_2\rangle$ is akin to <u>adding</u> angular momentum of unit j_1 to the state $|j_2, m_2\rangle$.

Wigner Eckart theorem :

To motivate the discussion, consider applying an electric field along the z-direction to a hydrogen atom. Using perturbation theory, one can show that the energy levels are shifted by an amount proportional to $\langle E_m | \hat{z} | E_n \rangle$ where $|E_n\rangle$ denotes eigenstates. Consider for example $|E_n\rangle = |2s \text{ state}\rangle$ i.e. principal quantum number $n=2$ and orbital angular momentum $l=0$. We ask: which $|E_m\rangle$ with principal quantum number $n=2$ can have a non-zero matrix element $\langle E_m | \hat{z} | 2s \text{ state} \rangle$? The answer is that $|E_m\rangle$ can be only $2p$ state with $m=0$ i.e. $l=1, m=0$.

To see this, we first notice that $\hat{z}$ is a 0-th component of a vector operator i.e. $\hat{z} = \hat{V}_0$ in our notation above. Therefore, $\hat{z}|l=0\rangle$ transforms like $|l=1, m=0\rangle \otimes |l=0, m=0\rangle$. Using angular momentum addition rules, this state transforms like $j=1$ and $m=0$, and therefore, $|E\,m\rangle$ must be the $|2p$ state$\rangle$ for $\langle E m|\hat{z}|2s$ state$\rangle$ to be non-zero.

Now we generalize this discussion in the form of the Wigner-Eckart theorem:

$$\langle \alpha', j\ m \,|\, \hat{O}_{j_1 m_1} \,|\, \alpha, j_2\, m_2\rangle$$
$$= \langle j\, m | j_1 j_2 ; m_1 m_2\rangle\, f(j, j_1, j_2, \alpha, \alpha')$$

where $\hat{O}_{j_1 m_1}$ is a tensor operator of rank

j_1, α, α' are some arbitrary quantum numbers other than the angular momentum, and f is some function of $j, j_1, j_2, \alpha, \alpha'$ that has <u>no</u> dependence on m, m_1 and m_2.

$\langle j\, m | j_1\, j_2 ; m_1\, m_2 \rangle$ is just the Clebsch - Gordon coefficient relating $|j_1 m_1\rangle \otimes |j_2, m_2\rangle$ to $|j\, m\rangle$. This theorem implies that the ratios such as

$$\frac{\langle \alpha', j\, m | \hat{O}_{j_1 m_1} | \alpha, j_2, m_2 \rangle}{\langle \alpha', j\, m' | \hat{O}_{j_1 m'_1} | \alpha, j_2, m'_2 \rangle}$$

$$= \frac{\langle j\, m | j_1\, j_2 ; m_1\, m_2 \rangle}{\langle j\, m' | j_1\, j_2 ; m'_1, m'_2 \rangle}$$

i.e. it is independent of α and α'!

<u>Proof</u> : We again consider the defining commutation relations for $\hat{O}_{jm}$, namely,

$$[\hat{J}_z, \hat{O}_{j,m}] = m\,\hat{O}_{j,m}$$

$$[\hat{J}_\pm, \hat{O}_{j,m}] = \sqrt{j(j+1) - m(m\pm 1)}\ \hat{O}_{j,m\pm 1}.$$

Consider the matrix element of the second eq. between $\langle \alpha', j' m' |$ and $|\alpha, j m\rangle \Rightarrow$

$$\langle \alpha', j\ m\ | J_\pm \hat{O}_{j_1 m_1} | j_2\ m_2\rangle$$

$$- \langle \alpha', j\ m\ | \hat{O}_{j_1 m_1} J_\pm\ |\alpha, j_2\, m_2\rangle$$

$$= \sqrt{j_1(j_1+1) - m_1(m_1\pm 1)}\langle \alpha', j\ m\ | \hat{O}_{j_1, m_1\pm 1}|\alpha, j_2 m_2\rangle$$

$$\Rightarrow \sqrt{j(j+1) - m(m\mp 1)}\ \langle \alpha', j, m\mp 1 | \hat{O}_{j_1 m_1} | \alpha\, j_2 m_2 \rangle$$

$$= \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)}\ \langle \alpha', j, m | \hat{O}_{j_1 m_1} | \alpha, j_2 m_2 \pm 1 \rangle$$

$$+ \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}\ \langle \alpha', j, m | \hat{O}_{j_1, m_1 \pm 1} | \alpha, j_2 m_2 \rangle$$

Interestingly, <u>exactly</u> the same equation is satisfied by the C-G coefficient

$\langle j, m | j_1\, j_2\, m_1\, m_2 \rangle$, where

$|j_1 j_2 m_1 m_2\rangle \equiv |j_1 m_1\rangle \otimes |j_2 m_2\rangle$.

To see this, we write

$$\langle j, m| = \sum_{m_1' m_2'} \langle j, m | j_1 j_2 m_1' m_2' \rangle \langle j_1 j_2 m_1' m_2' |$$

Applying $\hat{J}_\pm$ on both sides and taking the dot product with $|j_1 j_2 m_1 m_2\rangle$ implies,

$$\sqrt{j(j+1)-m(m\mp 1)} \quad \langle j, m\mp 1 | j_1 j_2 m_1 m_2 \rangle$$

$$= \sum_{m_1' m_2'} \langle j, m | j_1 j_2 m_1' m_2' \rangle \langle j_1 j_2 m_1' m_2' | \hat{J}_\pm | j_1 j_2 m_1 m_2 \rangle$$

(using $\hat{J}_\pm = \hat{J}_{1\pm} + \hat{J}_{2\pm}$) $\Rightarrow$

$$= \sum_{m_1' m_2'} \langle j, m | j_1 j_2 m_1' m_2' \rangle \langle j_1 j_2 m_1' m_2' | j_1 j_2, m_1 \pm 1, m_2 \rangle \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}$$

$$+$$

$$\sum_{m_1' m_2'} \langle j, m | j_1 j_2 m_1' m_2' \rangle \langle j_1 j_2 m_1' m_2' | j_1 j_2, m_1, m_2 \pm 1 \rangle \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)}$$

In the first term, the dot product $\Rightarrow m_1' = m_1 \pm 1$, $m_2' = m_2$. Similarly, in the second term, $m_1' = m_1$, $m_2' = m_2 \pm 1$ $\Rightarrow$

$$\sqrt{j(j+1)-m(m\mp 1)} \quad \langle j, m\mp 1 | j_1 j_2 m_1 m_2 \rangle$$

$$= \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)} \langle j, m | j_1 j_2 \, m_1 \pm 1, m_2 \rangle$$

$$+ \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)} \langle j, m | j_1 j_2 \, m_1, m_2 \pm 1 \rangle$$

This is <u>precisely</u> the same recursion relation as the one derived for the matrix element of the tensor operator $O_{j_1 m_1}$. Therefore, we have two sets of recursion relations, $\sum_j a_{ij} x_j = 0$ and $\sum_j a_{ij} y_j$ with $x_i \sim$ matrix elements of $\hat{O}_{jm}$ and $y_i \sim$ C-G coefficients. The only way to satisfy them is $x_i \propto y_i$ i.e. $x_i = c y_i$. This concludes the proof of the Wigner-Eckart theorem.