Hilbert Space of an Electron

The Hilbert space of electron is spanned by the basis states

$$\left\{ |\vec{r}\rangle \otimes |j=\tfrac{1}{2}, \hat{j}_z = \tfrac{1}{2}\rangle, \right.$$
$$\left. |\vec{r}\rangle \otimes |j=\tfrac{1}{2}, \hat{j}_z = \tfrac{1}{2}\rangle \right\}$$

where $|\vec{r}\rangle$ denotes a ket corresponding to the positional degrees of freedom while $|j=1/2, \hat{j}_z = \pm\frac{1}{2}\rangle$ denotes the two-dimensional vector space corresponding to spin degrees of freedom. Any operator acting on electron can always be decomposed into a <u>sum</u> of the form $\sum_i \hat{O}^i_r \otimes \hat{O}^i_s$ where $\hat{O}^i_r$ acts only on the positional degrees of freedom and $\hat{O}_s^i$ acts only on the spin degrees of freedom.

Any state corresponding to electron may be expanded as:

$$|\psi\rangle = \sum_{\vec{r}} \psi_+(\vec{r})\, |\vec{r}\rangle \otimes |j=1/2, j_z=\tfrac{1}{2}\rangle$$

$$+ \sum_{\vec{r}} \psi_-(\vec{r})\, |\vec{r}\rangle \otimes |j=\tfrac{1}{2}, j_z=-\tfrac{1}{2}\rangle$$

Therefore, one may write the wavefunction $\langle \vec{r}|\psi\rangle$ as a two-component object

$$\langle \vec{r}|\psi\rangle \equiv \begin{bmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{bmatrix} \begin{array}{l} \rightarrow j_z = +\tfrac{1}{2} \\ \rightarrow j_z = -\tfrac{1}{2}. \end{array}$$

where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ denotes the state corresponding to $|j=1/2, j_z=\tfrac{1}{2}\rangle$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ denotes the state corresponding to $|j=\tfrac{1}{2}, j_z=-\tfrac{1}{2}\rangle$.

Generator of rotation :

Electron carries both the orbital angular momentum (with $j =$ arbitrary integer) and the internal angular momentum ('spin') with $j = 1/2$.

The total angular momentum is given by

$$\vec{J} = \vec{L} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{S}$$

where $\vec{L}$ denotes the orbital angular momentum and $\vec{S}$ denotes the internal spin. The rotation is generated as :

$$e^{i(\vec{L} \otimes \mathbb{1}) \cdot \hat{n}_L \phi} \; e^{i(\mathbb{1} \otimes \vec{S}) \cdot \hat{n}_S \theta} \; |\psi\rangle$$

Addition of Angular Momentum

Main question: Given two spins $\vec{S}_1$ and $\vec{S}_2$ transform in the j_1 and j_2 representation respectively (i.e. whose angular momentum matrices are $(2j_1+1)\times(2j_1+1)$ and $(2j_2+1)\times(2j_2+1)$ - dimensional), what are the eigenvectors of the sum $\vec{S} = \vec{S}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \vec{S}_2$.

By construction $[S_i, S_j] = i\varepsilon_{ijk} S_k$, therefore $\vec{S}$ is also an angular momentum operator. Let's label the eigenstates of $\vec{S}_1, \vec{S}_2, \vec{S}$ as $|j_1 m_1\rangle$, $|j_2 m_2\rangle$ and $|j\, m, j_1 j_2\rangle$ where j is as yet unknown. We want to find $|j\, m, j_1 j_2\rangle$ in terms of $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$.

Main result: The allowed values of j are $|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$, and for each allowed value of j, 'm' takes values $-j, -j+1, \ldots, +j$ as expected for an angular momentum.

Since $S_z = S_{1z} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes S_{2z}$

$\Rightarrow$ if $|j\, m, j_1\, j_2\rangle$

$$= \sum_{m_1, m_2} \langle j_1\, m_1| \otimes \langle j_2\, m_2| j\, m, j_1\, j_2\rangle$$

then only the values of m_1, m_2 that satisfy $m_1 + m_2 = m$ are allowed.

A basic check on the above result is that the size of the Hilbert space is preserved, as it should (since we are just changing the basis from $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$ to

$|j\, m; j_1 j_2\rangle$.

Size of Hilbert space using the original product basis $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$

$= (\#$ of allowed values of $m_1)$

$\times$ (" " " " m_2)

$= (2j_1+1) \times (2j_2+1)$.

Size of Hilbert space using the new basis $|j\, m, j_1\, j_2\rangle$

$=$ (assuming $j_1 > j_2$)

$$\sum_{i=0}^{2j_2} [2(j_1 - j_2 + i) + 1]$$

$$= (2j_2+1) + 2(j_1-j_2)(2j_2+1)$$

$$+ \frac{2 \quad 2j_2 \, (2j_2+1)}{2}$$

$$= (2j_2+1)[1 + 2j_1 - 2j_2 + 2j_2]$$

$$= (2j_1+1)(2j_2+1).$$

A shorthand notation for the main result is that

$$j_1 \otimes j_2 = \bigoplus_{i=0}^{2j_2} [|j_1-j_2|+i]$$

For example,

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

i.e. adding two spin-$1/2$ spins leads to a direct <u>sum</u> of a spin-0 and a spin-1 Hilbert space.

General procedure :

Given the task of adding spins j_1 and j_2, it always helps to start with the state $|j_1\; m_1 = j_1\rangle \otimes |j_2\; m_2 = j_2\rangle$ because there is a unique state of the total spin with $S_z = j_1 + j_2$, namely the state $|j_1 + j_2\; m = j_1 + j_2; j_1 j_2\rangle$

Therefore $|j_1 + j_2\; m = j_1 + j_2, j_1\; j_2\rangle$

$$= |j_1\; m_1 = j_1\rangle \otimes |j_2\; m_2 = j_2\rangle.$$

To obtain other states with $j = j_1 + j_2$ that have other values of allowed m values, one applies S_-

$= S_{-1} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes S_{-2}$ on both sides repeatedly. This gives all states with $j = j_1 + j_2$.

Next, consider $\hat{j} = \hat{j}_1 + \hat{j}_2 - 1$.

Now, the maximum value of m is $\hat{j}_1 + \hat{j}_2 - 1$. There are only two states in the product basis with this value of $m_1 + m_2 = m$.

$$|\hat{j}_1 + \hat{j}_2 - 1 \quad m = \hat{j}_1 + \hat{j}_2 - 1 \, , \, \hat{j}_1 \quad \hat{j}_2 \rangle$$

$$= A \, |\hat{j}_1 \quad m_1 = \hat{j}_1 \rangle \otimes |\hat{j}_2 \quad m_2 = \hat{j}_2 - 1 \rangle$$

$$+ B \, |\hat{j}_2 \quad m_1 = \hat{j}_1 - 1 \rangle \otimes |\hat{j}_1 \quad m_2 = \hat{j}_2 \rangle .$$

We have two constraints:

- This state must be orthogonal to $|\hat{j} = \hat{j}_1 + \hat{j}_2 \quad m = \hat{j}_1 + \hat{j}_2 - 1 , \hat{j}_1 \, \hat{j}_2 \rangle$

- $A^2 + B^2 = 1$. One can choose all coefficients to be real.

This yields A, B.

After applying S_- to both sides, one obtains the expressions for all states with $j = j_1 + j_2 - 1$.

One can then move onto $j = j_1 + j_2 - 2$ and obtain the highest m state using analogous procedure and keep repeating this whole procedure until one reaches $j = |j_1 - j_2|$.

Shankar's book reviews this procedure in full detail if it's not clear from these notes (we will do explicit calculations to illustrate it). The following identity is useful:

$$J_\pm |j\, m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j\, m\pm 1\rangle$$

Example 1: Adding two spin-1/2 spins

Following aforementioned result,

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 .$$

$$|j=1 \quad m=1\rangle = |j=\tfrac{1}{2} \quad m=\tfrac{1}{2}\rangle \otimes |j=\tfrac{1}{2} \quad m=\tfrac{1}{2}\rangle$$

To obtain $|j=1, m=0\rangle$ we act with $J_- = J_{-1} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes J_{-2}$ on both sides.

$$\Rightarrow \sqrt{2 - 1\times 0} \quad |j=1, \quad m=0\rangle$$

$$= \sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right) - \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)}$$

$$\Big[|j=\tfrac{1}{2} \quad m=-\tfrac{1}{2}\rangle \otimes |j=\tfrac{1}{2} \quad m=\tfrac{1}{2}\rangle$$

$$+ |j=\tfrac{1}{2} \quad m=\tfrac{1}{2}\rangle \otimes |j=\tfrac{1}{2} \quad m=-\tfrac{1}{2}\rangle$$

$\Rightarrow |j=1 \quad m=0\rangle$

$$= \frac{1}{\sqrt{2}}\Big[|j=\tfrac{1}{2} \;\; m=-\tfrac{1}{2}\rangle \otimes |j=\tfrac{1}{2} \;\; m=\tfrac{1}{2}\rangle$$

$$+ |j=\tfrac{1}{2} \;\; m=\tfrac{1}{2}\rangle \otimes |j=\tfrac{1}{2} \;\; m=-\tfrac{1}{2}\rangle \Big]$$

One can obtain the $|j=1, m=-1\rangle$ state either by acting with J_- on $|j=1 \;\; m=0\rangle$, or, more simply, by observation that there is a unique state in the product basis with $m = m_1 + m_2 = -1$.

$|j=1, \quad m=-1\rangle$

$= |j=\tfrac{1}{2} \;\; m=-\tfrac{1}{2}\rangle \otimes |j=\tfrac{1}{2} \;\; m=-\tfrac{1}{2}\rangle.$

Next, we move to $j=0$ state. There is only one state in this sector, namely $|j=0\ m=0\rangle$. It must be orthogonal to $|j=1, m=0\rangle$, and therefore, it is

$$|j=0\ m=0\rangle = \frac{1}{\sqrt{2}}\Big[\, |j=\tfrac{1}{2}\ m=-\tfrac{1}{2}\rangle \otimes |j=\tfrac{1}{2}\ m=\tfrac{1}{2}\rangle - |j=\tfrac{1}{2}\ m=\tfrac{1}{2}\rangle \otimes |j=\tfrac{1}{2}\ m=-\tfrac{1}{2}\rangle \Big]$$

Note the relative minus sign between the two kets.

This completes the construction for $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.

Example 2: Adding two spin-1 spins

$1 \otimes 1 = 0 \oplus 1 \oplus 2.$

We start with

$$|j=2\ m=2\rangle = |j=1\ m=1\rangle \otimes |j=1\ m=1\rangle$$

which is uniquely determined.

Applying J_- on both sides,

$$\sqrt{2\times 3 - 2\times 1}\ |j=2, m=1\rangle = \sqrt{1\times 2 - 1\times 0}\,\big[\,|j=1\ m=0\rangle \otimes |j=1\ m=1\rangle + |j=1\ m=1\rangle \otimes |j=1\ m=0\rangle\big]$$

$$\Rightarrow |j=2\ m=1\rangle = \frac{1}{\sqrt{2}}\big[\,|j=1\ m=0\rangle \otimes |j=1\ m=1\rangle + |j=1\ m=1\rangle \otimes |j=1\ m=0\rangle\big]$$

Applying J_- once more,

$$\sqrt{6}\ |j=2, m=0\rangle = \frac{\sqrt{2}}{\sqrt{2}}\big[\,|j=1\ m=-1\rangle \otimes |j=1\ m=1\rangle + |j=1\ m=0\rangle \otimes |j=1\ m=0\rangle + |j=1\ m=0\rangle \otimes |j=1\ m=0\rangle + |j=1\ m=1\rangle \otimes |j=1\ m=-1\rangle\big]$$

$\Rightarrow |j=2 \quad m=0\rangle$

$$= \frac{1}{\sqrt{6}}\Big[\, |j=1 \quad m=-1\rangle \otimes |j=1 \quad m=1\rangle$$
$$+ 2\,|j=1 \quad m=0\rangle \otimes |j=1 \quad m=0\rangle$$
$$+ |j=1 \quad m=1\rangle \otimes |j=1 \quad m=-1\rangle \Big]$$

By symmetry (or by applying J_-)

$|j=2 \quad m=-1\rangle$

$$= \frac{1}{\sqrt{2}}\Big[|j=1 \quad m=0\rangle \otimes |j=1\, m=-1\rangle + |j=1 \quad m=-1\rangle \otimes |j=1\, m=0\rangle \Big]$$

and $|j=2 \quad m=-2\rangle = |j=1 \quad m=-1\rangle \otimes |j=1 \quad m=-1\rangle$.

Next we consider $j=1$ sector.

$|j=1 \quad m=1\rangle$ must be orthogonal to $|j=2, m=1\rangle$. $\Rightarrow$

$|j=1 \quad m=\pm 1\rangle =$

$$\frac{1}{\sqrt{2}}\Big[|j=1 \quad m=\pm 1\rangle \otimes |j=1 \quad m=0\rangle - |j=1 \quad m=0\rangle\, |j=1 \quad m=\pm 1\rangle \Big]$$

One may now apply J_- on both sides to obtain $|j=1 \;\; m=0\rangle$

$$= \frac{1}{\sqrt{2}} \Big[|j=1 \;\; m=1\rangle \otimes |j=1 \;\; m=-1\rangle - |j=1 \;\; m=-1\rangle \otimes |j=1 \;\; m=1\rangle \Big]$$

Finally to obtain $|j=0 \;\; m=0\rangle$, it must be orthogonal to both $|j=2 \;\; m=0\rangle$ and $|j=1 \;\; m=0\rangle$.

One may obtain it as as 'cross-product' of these other two vectors which gaurantees orthonormality:

$|j=2 \;\; m=0\rangle =$

$$\begin{array}{ccc} |j=1 \;\; m=+1\rangle \otimes |j=1 \;\; m=-1\rangle & |j=1 \; m=0\rangle |j=1 \; m=0\rangle & |j=1 \; m=-1\rangle \otimes |j=1 \; m=1\rangle \\ \hline 1/\sqrt{6} & \frac{2}{\sqrt{6}} & 1/\sqrt{6} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{array}$$

$$= -\frac{1}{\sqrt{3}} |j=1 \ m=1\rangle \otimes |j=1 \ m=-1\rangle$$

$$+ \frac{1}{\sqrt{3}} |j=1 \ m=0\rangle \otimes |j=1 \ m=0\rangle$$

$$- \frac{1}{\sqrt{3}} |j=1 \ m=-1\rangle \otimes |j=1 \ m=1\rangle .$$

More generally, given $N-1$ orthonormal vectors $v_1, v_2, \ldots v_{N-1}$ that live in an N dimensional vector space, one can find a vector v_N that is orthogonal to all of them as

$$v_N = \det \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \cdots & \hat{e}_N \\ v_{1,1} & v_{1,2} & \cdots & v_{1,N} \\ \vdots & & & \\ v_{N-1,1} & v_{N-1,2} & \cdots & v_{N-1,N} \end{bmatrix}$$

where $\{\hat{e}_i\}$ span the vector space and $v_{j,\alpha}$ is the α'th component of the vector $\vec{v}_j$.