Coordinate representation of eigenfunctions of L^2 and L_z:

$\langle \theta, \varphi | l\, m \rangle = Y_{lm}(\theta, \varphi)$ where

Y_{lm} are spherical harmonics

General structure of $Y_{lm}(\theta, \varphi)$:

$$\langle \theta, \varphi | L_z | j\, m \rangle = m \langle \theta, \varphi | j, m \rangle$$

$$\Rightarrow \; -i \langle \theta, \varphi | \frac{\partial}{\partial \varphi} | j\, m \rangle = m \langle \theta, \varphi | j, m \rangle$$

$$\Rightarrow \; \langle \theta, \varphi | j, m \rangle = e^{im\varphi} \langle \theta, \varphi = 0 | j, m \rangle$$

This also implies that $Y_{lm}(\theta, \varphi)$ makes sense only for m, l integers.

Explicit form of Y_{lm} ?

Use $\hat{L}_+ |l\, l\rangle = 0$

$\Rightarrow \langle \theta, \varphi | \hat{L}_+ | l\, l \rangle = 0$

$\hat{L}_\pm$ as a differential operator is given by $L_\pm = \pm e^{\pm i\varphi}\left[\frac{\partial}{\partial\theta} \pm i \cot(\theta) \frac{\partial}{\partial\varphi}\right]$

Write $\langle \theta, \varphi | l\, l\rangle = f(\theta) e^{il\varphi}$

$$\Rightarrow \left[\frac{\partial f}{\partial\theta} - l \cot(\theta) f\right] = 0$$

$$\Rightarrow \quad f(\theta) = [\sin(\theta)]^l$$

$$\Rightarrow \psi(\theta, \varphi) \propto (\sin\theta)^l e^{\pm il\varphi}$$

upto normalization.

Since $|l\ l-1\rangle \propto L_-|l\ l\rangle$.

One may now find $Y^m{}_l(\theta,\varphi)\ \forall\ m$.

For example, $\langle\theta\varphi|L_\pm|l,m\rangle = L_\pm Y_{lm}(\theta,\varphi)$

$= \langle\theta,\varphi|\sqrt{l(l+1)-m(m\pm1)}\ |l,m\pm1\rangle$

$= \sqrt{l(l+1)-m(m\pm1)}\ \langle\theta,\varphi|l,m\pm1\rangle.$

$= \sqrt{l(l+1)-m(m\pm1)}\ Y_{l,m\pm1}.$

Let's consider an example with $l=1$.

$$Y_{11} \propto e^{i\varphi}\sin(\theta) = c\,e^{i\varphi}\sin(\theta)$$

where c is a constant that we will determine using normalization later.

To obtain Y_{10}, we apply the differential operator L_- as discussed above

$$L_-Y_{11} = -e^{-i\varphi}\left[\frac{\partial}{\partial\theta} - i\cot(\theta)\frac{\partial}{\partial\varphi}\right]Y_{11}$$

$$= -c \left[\cos(\theta) + \cos(\theta) \right] = -2c \cos(\theta)$$

$$= Y_{10} \sqrt{2} \Rightarrow Y_{10} = -\sqrt{2}\, c \cos(\theta)$$

To determine 'c', we use

$$\langle l\, m | l'\, m' \rangle = \delta_{ll'} \delta_{mm'}$$

$$= \int \langle l\, m | \theta \varphi \rangle \langle \theta \varphi | l'\, m' \rangle \, d\Omega$$

where $d\Omega = d\varphi \, d\cos\theta$ is an infinitesimal solid angle.

$$= \int Y_{lm}(\theta \varphi)\, Y_{l'm'}(\theta \varphi) \quad d(\cos\theta)\, d\varphi$$

for example, when $l = 1$ $m = 0$.

$$\int Y_{10}^2(\theta, \varphi) \; d(\cos\theta)\, d\varphi = 1$$

$$\Rightarrow \quad 2\, c^2 \int_{-1}^{1} \cos^2\theta \; d(\cos\theta) \times 2\pi = 1$$

$$\Rightarrow \quad 2 \times 2\pi \times \frac{2}{3}\, c^2 = 1 \Rightarrow c = \pm \sqrt{\frac{3}{8\pi}}$$

$$\Rightarrow \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos(\theta), \; Y_{1\pm1} = \mp \sqrt{\frac{3}{8\pi}}\, e^{\pm i\varphi} \sin(\theta)$$

Using the same technique, one finds, for $m > 0$,

$$Y_{lm}(\theta,\varphi) \sim e^{im\varphi}[\sin\theta]^{-m} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} [\sin(\theta)]^{2l}$$

(upto normalization).

while $Y_{l-m} = (-)^m (Y^m_l)^*$.

Heuristic "Picture" of $\langle \theta, \varphi | j, m \rangle$