Eigenvalues and Eigenvectors of Angular momentum.

Since L_α don't commute with each other, we can't diagonalize all of them at once. What is the maximal set of operators that can be diagonized?

Answer : $(L_i , \vec{L}^2)$ where i can be chosen to be $x, y,$ or z.

Check : $[L_i , L_x^2 + L_y^2 + L_z^2] = 0$.

Let's denote eigenvector as $|\alpha , \beta\rangle$

with $\quad L^2 |\alpha , \beta\rangle = \alpha |\alpha , \beta\rangle$ [$\Rightarrow \alpha > 0$]

$\quad L_z |\alpha , \beta\rangle = \beta |\alpha , \beta\rangle$.

Define raising and lowering operators (analogs of ladder operators a^+ , a in harmonic oscillator).

$$L_\pm = L_x \pm i L_y .$$

Check : $[L_z , L_\pm] = \pm L_\pm ,$

and $[L^2, L_\pm] = 0.$

Why are these called lowering/raising operators?

$$L_z \, L_+ |\alpha, \beta\rangle$$
$$= [L_+ + L_+ L_z] |\alpha, \beta\rangle$$
$$= L_+ |\alpha, \beta\rangle + \beta L_+ |\alpha, \beta\rangle$$
$$= (1+\beta) |\alpha, \beta\rangle$$

$\Rightarrow$ L_+ raises the L_z eigenvalue by 1. Similarly, $L_z L_- |\alpha, \beta\rangle = (\beta - 1) |\alpha, \beta\rangle$

Since $[L_+, L^2] = 0$, L_+ does not change L^2 eigenvalue: $L^2 L_+ |\alpha, \beta\rangle = \alpha |\alpha, \beta\rangle$.

Since $L^2 - L_z^2 = L_x^2 + L_y^2$ is a positive definite operator $\Rightarrow$ $\alpha \geq \beta^2$.

$\Rightarrow$ L_+ cannot keep raising L_z eigenvalue indefinitely. There must exist a state $|\alpha, \beta_{max}\rangle$ so that

$$L_+ |\alpha, \beta_{max}\rangle = 0.$$

$\Rightarrow \quad L_- L_+ |\alpha, \beta_{max}\rangle = 0$

But $L_- L_+ = (L_x - iL_y)(L_x + iL_y)$

$= L_x^2 + L_y^2 - L_z = L^2 - L_z^2 - L_z$

$\Rightarrow \quad \alpha - \beta_{max}^2 - \beta_{max} = 0$

$\Rightarrow \quad \alpha = \beta_{max}(\beta_{max} + 1).$

Similarly, L_- cannot indefinitely lower L_z eigenvalue due to the bound $\alpha \geq \beta^2$.

There must exist a state $|\alpha, \beta_{min}\rangle$ so that $L_- |\alpha, \beta_{min}\rangle = 0$. Using similar steps as above, $\alpha = \beta_{min}(\beta_{min} - 1)$.

Comparing the two expressions for

$\alpha = \beta_{max}(1 + \beta_{max}) = \beta_{min}(\beta_{min} - 1)$

$\Rightarrow \quad \beta_{min} = -\beta_{max}$ (the other possible soln. is not real).

The only constraint on β_{min} and β_{max} is that $\beta_{max} - \beta_{min} = 2\beta_{max}$ is an integer.

$\Rightarrow \quad \beta_{max} = \dfrac{\text{integer}}{2} = j$ with

$j = 0, \frac{1}{2}, 1, \frac{3}{2}$ etc.

$$\Rightarrow \quad \alpha = \beta_{max}(1 + \beta_{max})$$

$$= j(j+1)$$

$$= 0, \frac{3}{4}, 2, \text{ etc.}$$

From now on, let's denote β as m

$= j, j-1, j-2, \ldots\ldots, -j+1, -j.$

Thus, altogether, for a fixed j (i.e. fixed β_{max}), the Hilbert space has $2j+1$ orthonormal states $|j,m\rangle$ each of which correspond to a distinct m = eigenvalue of L_z.

$$L^2 \, |j, m\rangle = j(j+1) \, |j, m\rangle$$

$$L_z \, |j, m\rangle = m \, |j, m\rangle.$$

As we will soon see,
when $j =$ integer, L_α are standard angular momentum operators since L_z eigenvalues are integers, and therefore $\langle \theta, \varphi | j, m\rangle = e^{im\varphi} f(\theta)$ is single-valued (i.e. $e^{im\varphi} = e^{im(\varphi + 2\pi)}$)

However, when $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$ i.e. half-odd-integer, then L_α cannot be interpreted as a standard angular momentum operator since $e^{im\varphi}$ is not single-valued.

The only way to make sense of this case is to work with multi-component wavef^n eg. $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$. we will return to this later.

Block-diagonal structure of $\vec{J}$

Above we determined $\langle j\, m | L_z | j'\, m' \rangle$ and $\langle j\, m | L^2 | j'\, m' \rangle$. Lets now also determine $\langle j\, m | L_x | j'\, m' \rangle$ and $\langle j\, m | L_y | j'\, m' \rangle$

Based on above discussion,

$$L_\pm | j\, m \rangle = C_\pm(j, m) | j, m \pm 1 \rangle$$

$$\Rightarrow \langle j'\, m' | L_x | j\, m \rangle = f_x(j\, m)\, \delta_{jj'}$$

Similarly for L_y.

$$\Rightarrow L_{x,y} = \begin{bmatrix} \boxed{\begin{matrix} 1\times 1 \\ j=0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 2\times 2 \\ j=1/2 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 3\times 3 \\ j=1 \end{matrix}} & \\ & & & \boxed{\begin{matrix} 4\times 4 \\ j=3/2 \end{matrix}} \end{bmatrix}$$

Block-diagonal structure.

etc.

The commutation relations $[L_\alpha, L_\beta] = i\varepsilon_{\alpha\beta\gamma} L_\gamma$ are satisfied by each block separately.

i.e. $[L_\alpha^{(j)}, L_\beta^{(j')}] = i\varepsilon_{\alpha\beta\gamma}\, \delta_{jj'}\, L_\gamma^{(j)}$

"Physical meaning" of different blocks:

They correspond to different <u>irreducible</u> <u>representations</u> of angular momentum.

Vector operators act like J_+, J_-:

Consider a vector operator $\vec{V}$ i.e. an operator that satisfies $[\hat{V}_i, \hat{J}_j] = i\varepsilon_{ijk}\hat{V}_k$

$\Rightarrow \quad [V_x, J_z] = -iV_y, \quad [V_y, J_z] = iV_x$

$\Rightarrow \quad [V_x + iV_y, J_z] = -(V_x + iV_y)$

$\Rightarrow \quad [J_z, V_+] = V_+, \quad [J_z, V_-] = -V_-$

Consider the state $V_+|j,m\rangle$

$J_z V_+|j,m\rangle = (V_+ + V_+ J_z)|j,m\rangle$

$= (m+1)(V_+|j,m\rangle) \Rightarrow V_+$ acts like J_+. Similarly V_- acts like J_-.

Explicit form of J_i matrices of various j

From now on, let's denote L_i as J_i, as is more conventional (L_i typically refers to $j=$ integer, while J_i refers to both $j=$ integer and $j=\frac{1}{2}\times$ odd integer).

To figure out the matrices corresponding to J_i, we need the matrix elements $\langle j,m'|J_i|j,m\rangle$ for a fixed j. To do so, we first need the numbers c_+, c_- in

$$J_\pm |j,m\rangle = c_\pm(j,m)\,|j,m\pm 1\rangle$$

To obtain $c_\pm$, we take the overlap of $J_\pm|j,m\rangle$ with itself:

$$\langle j,m|\,J_\mp \; J_\pm|j,m\rangle = |c_\pm(j,m)|^2$$

(we are assuming that the states $|j,m\rangle$ are already normalized).

Now, $\langle j, m | J_{\mp} J_{\pm} | j, m \rangle$

$$= \langle j, m | (J_x \mp i J_y)(J_x \pm i J_y) | j, m \rangle$$

$$= \langle j, m | J_x^2 + J_y^2 \mp i [J_y, J_x] | j, m \rangle$$

$$= \langle j, m | J^2 - J_z^2 \mp J_z | j, m \rangle$$

$$= j(j+1) - m^2 \mp m$$

$$\Rightarrow \quad C_{\pm}(j, m) = \sqrt{j(j+1) - m(m \pm 1)}$$

upto an irrelevant phase factor that we set to 1.

$$\Rightarrow \quad J_{\pm} | j, m \rangle = \sqrt{j(j+1) - m(m \pm 1)} \; | j, m \pm 1 \rangle$$

Combined with $J_z | j, m \rangle = m | j, m \rangle$, we can now determine the matrix elements $\langle j, m' | J_x | j, m \rangle$, $\langle j, m' | J_y | j, m \rangle$, $\langle j, m' | J_z | j, m \rangle$.

$$\langle j, m | J_z | j, m \rangle = m \, \delta_{m,m'}$$

$$\langle j, m' | J_\pm | j, m \rangle = \sqrt{j(j+1) - m(m \pm 1)} \, \delta_{m', m\pm 1}$$

$\Rightarrow$

$$\langle j, m' | J_x | j, m \rangle$$

$$= \frac{1}{2} \left[\sqrt{j(j+1) - m(m+1)} \, \delta_{m', m+1} + \sqrt{j(j+1) - m(m-1)} \, \delta_{m', m-1} \right]$$

$$\langle j, m' | J_y | j, m \rangle$$

$$= \frac{1}{2i} \left[\sqrt{j(j+1) - m(m+1)} \, \delta_{m', m+1} - \sqrt{j(j+1) - m(m-1)} \, \delta_{m', m-1} \right]$$

For example, for $j = 1/2$: $J_i = \frac{\sigma_i}{2}$ [σ_i are the Pauli matrices]

while for $j = 1$,

$$J_x = \frac{1}{2} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}, \quad J_y = \frac{i}{2} \begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$J_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

as one may verify from the above derived general expressions.