1 Angular momentum as a generator of rotations

Recall that the operator $\hat{T}_a = e^{-\mathrm{i}\hat{p}a}$ generates translations, i.e., $\hat{T}_a |x\rangle = |x + a\rangle$. Inspired by this, now we will define an operator $D(R)$ that generates a rotation R. To specify a rotation, one needs to specify the axis $\hat{n}$ around which the rotation is being performed and an angle θ that quantifies the amount of rotation performed around $\hat{n}$.

Similar to the case of linear momentum where we first focussed on infinitesimal translations, it's again easier to consider infinitesimal rotations. Under an infinitesimal rotation $\delta\theta$ along $\hat{n}$, using elementary geometry, you may show that a three-dimensional vector $\vec{r}$ (not an operator) transforms as

$$\vec{r} \to R(\vec{r}) = \vec{r} + \delta\theta\hat{n} \times \vec{r}$$

For example, when $\hat{n} = \hat{z}$, the rotated vector is (to linear order in $\delta\theta$):

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \to \vec{r}' = \delta\theta\hat{z} \times (x\hat{x} + y\hat{y}) = (x - y\delta\theta)\hat{x} + (y + x\delta\theta)\hat{y} + z\hat{z}$$

We would like to define a corresponding unitary operator $D(R_{\hat{n}}, \delta\theta)$ so that

$$D(R_{\hat{n}}, \delta\theta) |\vec{r}\rangle = |\vec{r}'\rangle$$

where $\vec{r}'$ is the aforementioned rotated vector.

Claim: $D(R_{\hat{n}}, \delta\theta) = e^{-\mathrm{i}\delta\theta\hat{n}\cdot(\hat{x}\times\hat{p})}$ where $\hat{x}, \hat{p}$ are position and momentum operators. Note that $\hat{n}$ is just the unit vector specifying the axis of the rotation and is not an operator.

Verification:

$$\begin{aligned}
D(R_{\hat{n}}, \delta\theta) |\vec{r}\rangle &= (1 - \mathrm{i}\delta\theta\hat{n} \cdot (\hat{x} \times \hat{p})) |\vec{r}\rangle \\
&= |\vec{r}\rangle - \mathrm{i}\delta\theta(\hat{n} \times \vec{r}) \cdot \hat{p} |\vec{r}\rangle \\
&= |\vec{r}\rangle + \delta\theta(\hat{n} \times \vec{r}) \cdot \frac{\partial}{\partial\vec{r}} |\vec{r}\rangle \\
&= |\vec{r} + \delta\theta\hat{n} \times \vec{r}\rangle
\end{aligned}$$

(recall $|x + da\rangle = |x\rangle - da\frac{\partial}{\partial x} |x\rangle$)

Again, similar to the case of linear momentum, one may write an operator equation:

$$D(R_{\hat{n}}, \delta\theta)^\dagger \hat{r} D(R_{\hat{n}}, \delta\theta) = \hat{r} + \delta\theta\hat{n} \times \hat{r}$$

Again, keep in mind that on the RHS, $\hat{r}$ is an operator, while $\hat{n}$ is just a unit vector.

By applying repeated infinitesimal rotations around the same axis $\hat{n}$, the above equations hold also for finite rotations, i.e.

$$D(R_{\hat{n}}, \theta) |\vec{r}\rangle = |R(\vec{r})\rangle$$

and

$$D(R_{\hat{n}}, \theta)^\dagger \hat{r} D(R_{\hat{n}}, \theta) = R(\hat{r})$$

where $D(R_{\hat{n}}, \theta) = e^{-\mathrm{i}\theta\hat{n}\cdot(\hat{x}\times\hat{p})}$ and $R(\vec{r})$ is the rotated vector whose precise action can be determined using geometry (it is not $\vec{r} + \theta\hat{n} \times \vec{r}$).

2 Commutation relations for orbital angular momentum

As discussed above, the angular momentum operator is $\vec{L} = \hat{r} \times \hat{p}$ e.g. $L_x = y\hat{p}_z - z\hat{p}_y$.

Consider $[L_x, L_y]$:

$$\begin{aligned}
[L_x, L_y] &= [y\hat{p}_z - z\hat{p}_y, z\hat{p}_x - x\hat{p}_z] \\
&= y[p_z, z]p_x + z[z, p_x]p_y \\
&= \mathrm{i}(yp_x - xp_y) = \mathrm{i}L_z
\end{aligned}$$

Similarly, $[L_y, L_z] = \mathrm{i}L_x$, $[L_z, L_x] = \mathrm{i}L_y$. We will soon show that these commutation relations are rather constraining and determine the spectrum of $\vec{L}^2$. Further, there exist an infinite number of possibilities for matrices that satisfy $[J_i, J_j] = \mathrm{i}\hbar\epsilon_{ijk}J_k$ where each choice corresponds to a different sized matrix. Here are two possibilities:

2×2: $J_i = \frac{1}{2}\sigma_i$ where σ_i are the Pauli matrices.

3×3 matrices: $J_i = L_i$ where L_i are matrices

$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}$ $L_y = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}$, $L_z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (We will soon derive these)

2.1 Alternative perspective on angular momentum commutation relations

It is an important fact that even classically, rotation matrices in $d = 3$ don't commute:

$$R_x(\theta_x)R_y(\theta_y) - R_y(\theta_y)R_x(\theta_x) \neq 0$$

In fact, for infinitesimal θ_x, θ_y,

$$[R_x(\theta_x), R_y(\theta_y)] \sim R_z(\theta_x\theta_y)$$

Explicitly as matrices:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \approx 1 - \mathrm{i}\theta I_x$$

where $I_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}$ and we have assumed $|\theta| \ll 1$.

Similarly, $I_y = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}$, $I_z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

One notices $[I_x, I_y] = \mathrm{i}I_z$:

$$\begin{aligned}
[I_x, I_y] &= I_x I_y - I_y I_x \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \\
&= \mathrm{i} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{i}I_z
\end{aligned}$$

As discussed above, the unitary operator that implements the rotation is $D(R_{\hat{n}}, \theta) = e^{-\mathrm{i}\theta\hat{n}\cdot\vec{L}}$ where $\vec{L}$ is the quantum mechanical operator corresponding to the angular momentum.

We expect the operator $D(R_{\hat{n}}, \theta)$ to satisfy the same algebra as the 'classical' rotation matrices R_x, R_y, R_z:

$$[D(R_x, \delta\theta_x), D(R_y, \delta\theta_y)] - [D(R_y, \delta\theta_y), D(R_x, \delta\theta_x)] = -\mathrm{i}\delta\theta_x\delta\theta_y L_z$$

$$\Rightarrow [L_x, L_y] = \mathrm{i}L_z \text{ etc.}$$

i.e. $[L_i, L_j] = \mathrm{i}\epsilon_{ijk}L_k$, where ϵ_{ijk} is the fully anti-symmetric tensor.

3 Scalar Vs Vector Operators

In this section, we will denote angular momentum operator as $\vec{J}$ instead of $\vec{L}$. One defines a 'vector operator' as an operator satisfies the following commutation relations with the angular momentum operators:

$$[V_i, J_j] = \mathrm{i}\epsilon_{ijk}V_k$$

You may check that $\vec{r}$, $\vec{p}$, $\vec{L}$ are all vector operators.

We will now show that vector operators satisfy:

$$\begin{aligned}
e^{-\mathrm{i}\theta J_z} V_x e^{\mathrm{i}\theta J_z} &= V_x \cos\theta - V_y \sin\theta \\
e^{-\mathrm{i}\theta J_z} V_y e^{\mathrm{i}\theta J_z} &= V_x \sin\theta + V_y \cos\theta
\end{aligned}$$

Therefore, a vector operator transforms like a standard vector under rotation, hence the name.

To derive the above result, we start with

$$V_x(\theta) = e^{-\mathrm{i}\theta J_z} V_x e^{\mathrm{i}\theta J_z}$$

and then take derivative with respect to θ:

$$\frac{d}{d\theta}V_x(\theta) = \mathrm{i}e^{-\mathrm{i}\theta J_z}[J_z, V_x]e^{\mathrm{i}\theta J_z} = \mathrm{i}e^{-\mathrm{i}\theta J_z}\left(\mathrm{i}V_y\right)e^{\mathrm{i}\theta J_z} = -V_y(\theta)$$

Similarly, $\frac{d}{d\theta}V_y(\theta) = V_x(\theta)$. Note that these can also be thought of as Heisenberg's E.O.M. with θ = time, and J_z = Hamiltonian. Solving these two equations with the boundary condition $V_x(0) = V_x$, $V_y(0) = V_y$ gives the above expression of $V_x(\theta), V_y(\theta)$.

In contrast to vector operators, 'scalar operators' S transform trivially under rotations:

$$[S, J_i] = 0$$

An example is $S = V_i V_i$ where V_i is a vector operator (heuristically, S is the squared magnitude of $\vec{V}$ and hence does not change as $\vec{V}$ is rotated).

Check $[S, J_k] = [V_i V_i, J_k] = [V_i, J_k]V_i + V_i[V_i, J_k] = \mathrm{i}\hbar\epsilon_{kij}V_j V_i + \mathrm{i}\hbar\epsilon_{kij}V_i V_j = 0$ (recall that ϵ_{ijk} is fully antisymmetric).

4 Precession of a quantum-mechanical top

Let's first recall the precession of a classical top of mass m.

$$H = \frac{L^2}{2I} - mgz \quad (z = \text{height of the center of mass})$$

L_z is conserved since the torque is in the x-y plane $\Rightarrow$ the polar angle θ is independent of time, and only the azimuthal angle ϕ changes as a function of time.

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F} = -r\sin\theta\hat{\phi} \times mg\hat{z} = mgr\sin\theta\hat{\theta}$$

$L_x = L\sin\theta\cos\phi(t)$, $L_y = L\sin\theta\sin\phi(t)$

$$\frac{dL_x}{dt} = -L\sin\theta\sin\phi\frac{d\phi}{dt} = -mgr\sin\theta\sin\phi$$

$$\frac{dL_y}{dt} = L\sin\theta\cos\phi\frac{d\phi}{dt} = mgr\sin\theta\cos\phi$$

$$\Rightarrow \frac{d\phi}{dt} = \frac{mgr}{L} = \omega$$

$L_z = L\cos\theta \Rightarrow mgz = mgr\cos\theta = (mgr)\frac{L_z}{L}$

Therefore, one may write,

$$H = \frac{L^2}{2I} + \omega L_z \quad [\text{note that this is time-independent}]$$

and $\frac{d\vec{L}}{dt} = \omega\hat{z} \times \vec{L}$

$$\Rightarrow L_x(t) = L_x(t=0)\cos(\omega t) - L_y(t=0)\sin(\omega t)$$

$$L_y(t) = L_y(t=0)\cos(\omega t) + L_x(t=0)\sin(\omega t)$$

These classical results motivate us to define a quantum mechanical top via the Hamiltonian $H = \frac{\vec{J}^2}{2I} - \vec{\mu}\cdot\vec{B}$

One may choose $\vec{B}$ along the z-direction. So that $H = \frac{\vec{J}^2}{2I} + \omega J_z$ ($\omega = \frac{\mu B}{\hbar}$)

$[J^2, H] = [J_z, H] = 0 \Rightarrow J_z, J^2$ are constants of motion, similar to the classical case.

Heisenberg's Eq. of motion:

$$J_x(t) = e^{iHt/\hbar} J_x e^{-iHt/\hbar} = e^{i\omega J_z t} J_x e^{-i\omega J_z t}$$

Using above discussion of vector operators:

$$J_x(t) = J_x(0)\cos(\omega t) - J_y(0)\sin(\omega t)$$

$$J_y(t) = J_y(0)\cos(\omega t) + J_x(0)\sin(\omega t)$$

The derivation only relies on J being a vector operator i.e. $[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k$. Note the close parallel between the classical and the quantum mechanical top.

5 Wave-fn approach to precession for a spin-1/2 system

Let's specialize to 2×2 angular momentum matrices, so that $\vec{J} = \frac{1}{2}\vec{\sigma}$. Above we considered the operator approach to precession, where to calculate $\langle J_x(t)\rangle$ we evaluated $\langle\psi|e^{iHt}J_x e^{-iHt}|\psi\rangle$. One can also consider the Schrödinger approach where we instead evaluate $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$ and calculate $\langle\psi(t)|J_x|\psi(t)\rangle$.

At $t = 0$, let's say $\langle J_z\rangle = \frac{1}{2}\cos\theta$, $\langle J_x\rangle = \frac{1}{2}\sin\theta\cos\phi$, $\langle J_y\rangle = \frac{1}{2}\sin\theta\sin\phi$ i.e. $\langle\psi(0)|\vec{\sigma}|\psi(0)\rangle = \hat{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$

The corresponding state at $t = 0$ is the eigenvector of $\vec{\sigma}\cdot\hat{n}$ with eigenvalue +1. Using our earlier discussion:

$$|\psi(0)\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix}$$

Check:

$$\begin{aligned}
\langle\sigma_z\rangle &= \cos^2(\theta/2) - \sin^2(\theta/2) = \cos\theta \\
\langle\sigma_x\rangle &= 2\cos(\theta/2)\sin(\theta/2)\cos\phi = \sin\theta\cos\phi \\
\langle\sigma_y\rangle &= 2\cos(\theta/2)\sin(\theta/2)\sin\phi = \sin\theta\sin\phi
\end{aligned}$$

Therefore $|\psi(0)\rangle$ is indeed the correct initial state.

Now, we time evolve $|\psi(0)\rangle$ with respect to $H = \omega J_z = \frac{\omega}{2}\sigma_z$:

$$\begin{aligned}
|\psi(t)\rangle &= e^{-iHt}|\psi(0)\rangle \\
&= e^{-i\omega t\sigma_z/2}\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta/2)e^{-i\omega t/2} \\ \sin(\theta/2)e^{i(\phi+\omega t/2)} \end{pmatrix}
\end{aligned}$$

Therefore, under time-evolution, the state at time t is obtained from the state at time $t = 0$ by the substitution $\theta \to \theta$, $\phi \to \phi + \omega t$ i.e. $\langle J_z\rangle$ remains unchanged, while the vector $(\langle J_x\rangle, \langle J_y\rangle)$ rotates at an angular velocity ω, exactly in agreement with the result in the operator (Heisenberg) picture discussed in the last section.