

Composite Hilbert spaces

Ex. 1. Consider two distinguishable particles in a harmonic potential :

Classically,
$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\omega^2 x_2^2$$

Quantum mechanically, one needs to first define the Hilbert space for this composite system. For a single-particle, one choice of basis is $|x\rangle$. Another choice is $|n\rangle$ where $n = 0, 1, \dots, \infty$ denotes eigenvalue of $a^\dagger a$.

For the two-particle system, one may choose the basis to be

$|x_1\rangle \otimes |x_2\rangle$. Another possibility

is $|n_1\rangle \otimes |n_2\rangle$. Here $\otimes$ denotes

tensor product of states whose meaning is as follows. Consider an operator $\hat{O}_1$ that only acts on particle #1. Its action on $|x_1\rangle \otimes |x_2\rangle$ is defined as

$$\hat{O}_1 (|x_1\rangle \otimes |x_2\rangle) = \hat{O}_1 |x_1\rangle \otimes |x_2\rangle.$$

Therefore, in the composite Hilbert space, $\hat{O}_1$ may be written as $\hat{O}_1 \otimes \mathbb{1}_2$ where $\mathbb{1}_2$ denotes the identity operator acting on particle #2. Similarly, an operator $\hat{O}_2$ that acts only on particle #2 is more precisely written as $\mathbb{1}_1 \otimes \hat{O}_2$, and acts as

$$\mathbb{I}_1 \otimes \hat{O}_2 |x_1\rangle \otimes |x_2\rangle \\ = |x_1\rangle \otimes \hat{O}_2 |x_2\rangle.$$

More generally one can consider an operator $\hat{O}_1 \otimes \hat{O}_2$ that acts as $\hat{O}_1 \otimes \hat{O}_2 (|x_1\rangle \otimes |x_2\rangle)$

$$= \hat{O}_1 |x_1\rangle \otimes \hat{O}_2 |x_2\rangle.$$

Entangled states :

Above, we mentioned one possible choice of the basis states of the form

$|x_1\rangle \otimes |x_2\rangle$ or $|m_1\rangle \otimes |n_2\rangle$. Both of these have **direct product** form $|\phi_1\rangle \otimes |\phi_2\rangle$ where $|\phi_1\rangle$ lives in the Hilbert space belonging to particle 1, and similarly

$|\phi_2\rangle$ belongs to the Hilbert space belonging to particle 2. One may also construct states that do not have this form. For example, consider the state that still lives in the Hilbert space belonging to two particles in a harmonic potential:

$$|\psi\rangle = |n_1=0\rangle \otimes |n_2=1\rangle + |n_1=1\rangle \otimes |n_2=0\rangle$$

where n_1, n_2 again denote the occupation numbers $a_1^\dagger a_1$ and $a_2^\dagger a_2$ respectively.

One can show that there exists no choice of basis states in 1 and 2

so that $|\psi\rangle$ may be written as $|\phi_1\rangle \otimes |\phi_2\rangle$.

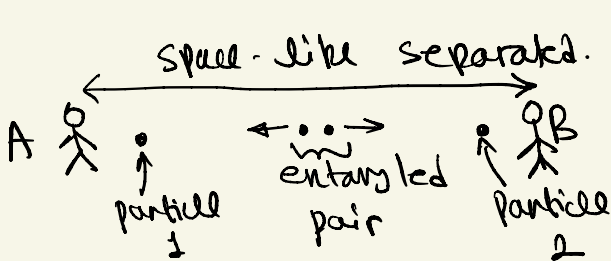
In other words, there does not exist any unitaries U_1, U_2

acting on the first and the second particles respectively such that

$$U_1 \otimes U_2 |\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle.$$

Such states, which have a non-direct-product form in any choice of basis are called entangled and they can have very counter-intuitive properties with no classical analog, as we will discuss soon.

EPR experiment with $|\psi\rangle = |n_1=0\rangle \otimes |n_2=1\rangle + |n_1=1\rangle \otimes |n_2=0\rangle$



Record of Measurement outcomes:

person A	person B
$n_1 = 0$	$n_2 = 1$
$n_1 = 1$	$n_2 = 0$

Perfect anti-correlation but still no faster than light communication because they would need to compare their records to notice the perfect anti-correlation!

Eigenstates and Eigenvectors of two particles in a harmonic trap:

Based on the above discussion, the Hamiltonian $H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\omega^2 x_2^2$ can more precisely be written as

$$H = \left(\frac{P_1^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 \right) \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \left(\frac{P_2^2}{2m} + \frac{1}{2}m\omega^2 x_2^2 \right)$$

For brevity, the tensoring with $\mathbb{1}_1$ or $\mathbb{1}_2$ is generally omitted.

The above Hamiltonian has a **separable** form i.e. $\hat{O}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \hat{O}_2$.

An example of a non-separable Hamiltonian would be above $H + x_1 \otimes x_2$.

Since H is separable, one can diagonalize its two pieces separately:

$$H = \omega \left[(a_1^\dagger a_1 + \frac{1}{2}) \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes (a_2^\dagger a_2 + \frac{1}{2}) \right]$$

where $a_1 = \sqrt{\frac{m\omega}{2}} x_1 + i \sqrt{\frac{1}{2m\omega}} p_1$

and $a_2 = \sqrt{\frac{m\omega}{2}} x_2 + i \sqrt{\frac{1}{2m\omega}} p_2$

are the usual annihilation operators

that satisfy, $[a_1, a_1^\dagger] = 1$,

$[a_2, a_2^\dagger] = 1$. Since a_1, a_2

act on different Hilbert spaces,

$[a_1, a_2] = [a_1, a_2^\dagger] = 0$.

Therefore, the eigenstates of H

are simply given by,

$$|n_1\rangle \otimes |n_2\rangle = (a_1^\dagger)^{n_1} |0\rangle_1 \otimes (a_2^\dagger)^{n_2} |0\rangle_2$$

$$= (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} |0\rangle_1 \otimes |0\rangle_2.$$

where $|0\rangle_1, |0\rangle_2$ are the respective vacuum states for the two oscillators.

The corresponding eigenvalue is

indeed $n_1 + \frac{1}{2} + n_2 + \frac{1}{2} = n_1 + n_2 + 1.$

Eigenstates in a real-space basis :

$$\langle x_1 | \otimes \langle x_2 | | n_1 \rangle \otimes | n_2 \rangle \equiv \psi_{n_1, n_2}(x_1, x_2)$$

$$= \langle x_1 | n_1 \rangle \langle x_2 | n_2 \rangle$$

$$\propto e^{-\frac{m\omega(x_1^2 + x_2^2)}{2}} H_{n_1}(\sqrt{m\omega} x_1) H_{n_2}(\sqrt{m\omega} x_2)$$

where H_n denotes the n -th Hermite polynomial.

Composite Hilbert Spaces made out of Hilbert spaces with finite dimension

Let's consider Hilbert space spanned by tensor product of the Hilbert spaces corresponding to two spin- $\frac{1}{2}$ objects.

One possible basis choice is

$$|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle.$$

Let's study the eigenstates of two different Hamiltonians that act on this Hilbert space.

$$(a) \quad H = Z_1 \otimes Z_2.$$

$$(b) \quad H = X_1 \otimes X_2 + Y_1 \otimes Y_2.$$

The Hilbert space is $2 \times 2 = 4$ -dimensional in both cases.

$$(a) H = Z_1 \otimes Z_2$$

The four eigenstates are $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$ and $|1\rangle \otimes |1\rangle$ with eigenvalues $1, -1, -1, 1$ respectively.

$$(b) H = X_1 \otimes X_2 + Y_1 \otimes Y_2.$$

One notices that $[Z_1 \otimes Z_2, H] = 0$.

Therefore, let us decompose the Hilbert space into direct sum of two subspaces where the states in the first subspace satisfy $Z_1 \otimes Z_2 |\phi\rangle = +1 |\phi\rangle$ and the states in the second subspace satisfy $Z_1 \otimes Z_2 |\phi\rangle = -1 |\phi\rangle$ [since $(Z_1 \otimes Z_2)^2 = 1 \otimes 1$, ± 1 are the only two eigenvalues of $Z_1 \otimes Z_2$]

The basis for the first subspace may be chosen as $\{ |0\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle \}$

and that for the second subspace as

$\{ |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle \}$. Note that

$$Z_1 \otimes Z_2 [c_1 |0\rangle \otimes |0\rangle + c_2 |1\rangle \otimes |1\rangle]$$

$$= +1 [c_1 |0\rangle \otimes |0\rangle + c_2 |1\rangle \otimes |1\rangle]$$

irrespective of c_1, c_2 . Similarly for

the second subspace. Therefore,

we only need to diagonalize the

2×2 matrices corresponding to

each subspace separately.

(i) Subspace 1:

$$(X_1 \otimes X_2 + Y_1 \otimes Y_2) (|0\rangle \otimes |0\rangle)$$

$$= |1\rangle \otimes |1\rangle - |1\rangle \otimes |1\rangle = 0$$

$$(X_1 \otimes X_2 + Y_1 \otimes Y_2) (|1\rangle \otimes |1\rangle)$$

$$= |0\rangle \otimes |0\rangle - |0\rangle \otimes |0\rangle = 0$$

Therefore the Hamiltonian within this subspace is just a 2×2 matrix with all zero entries:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

And the eigenvectors may be chosen as any two independent vectors within this subspace e.g. $|0\rangle \otimes |0\rangle$ and $|1\rangle \otimes |1\rangle$.
Or even $|0\rangle \otimes |0\rangle \pm |1\rangle \otimes |1\rangle$.

(ii) subspace 2:

$$\begin{aligned} & (x_1 \otimes x_2 + y_1 \otimes y_2) |0\rangle \otimes |1\rangle \\ &= |1\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle = 2 |1\rangle \otimes |0\rangle. \end{aligned}$$

$$\begin{aligned} & (x_1 \otimes x_2 + y_1 \otimes y_2) |1\rangle \otimes |0\rangle \\ &= 2 |0\rangle \otimes |1\rangle \end{aligned}$$

Therefore, the restriction of H to this subspace is $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$. We already

know the eigenvectors and eigenvalues of this matrix (since its just like the Pauli matrix X).

Therefore, eigenvectors and eigenvalues within the second subspace:

$$\text{eigenvalue} = +2$$

$$\text{eigenvector} = \frac{1}{\sqrt{2}} [10\rangle \otimes 11\rangle + 11\rangle \otimes 10\rangle]$$

$$\text{eigenvalue} = -2$$

$$\text{eigenvector} = \frac{1}{\sqrt{2}} [10\rangle \otimes 11\rangle - 11\rangle \otimes 10\rangle].$$

Therefore the four eigenvalues of H are $0, 0, \pm 2$ and the corresponding eigenvectors are $10\rangle \otimes 10\rangle$, $11\rangle \otimes 11\rangle$, $\frac{1}{\sqrt{2}} [10\rangle \otimes 11\rangle \pm 11\rangle \otimes 10\rangle]$.

↑
entangled (EPR/Bell) pair

Explicit form of H :

$$X_1 \otimes X_2 + Y_1 \otimes Y_2$$

$|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle$

$$= \begin{matrix} \begin{matrix} |0\rangle|0\rangle \\ |0\rangle|1\rangle \\ |1\rangle|0\rangle \\ |1\rangle|1\rangle \end{matrix} \end{matrix} \begin{bmatrix} & & & \\ & X_2 & & \\ & & & \\ X_2 & & & \end{bmatrix} + \begin{bmatrix} & & & \\ & & & -iY_2 \\ & & & \\ +iY_2 & & & \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & \\ & 1 & 0 & \\ 0 & 1 & & 0 \\ 1 & 0 & & \end{bmatrix} + \begin{bmatrix} 0 & & & -1 \\ & 1 & & \\ -1 & 1 & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

GHZ Experiment

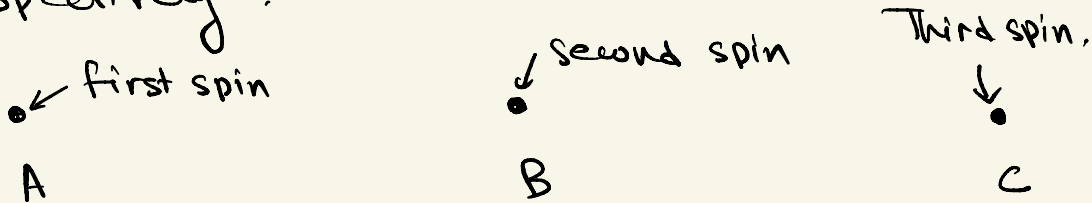
$$|\psi\rangle =$$

Consider the state $\frac{1}{\sqrt{2}}[|0\rangle \otimes |0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle \otimes |1\rangle]$

that lives in the Hilbert space of three spin- $1/2$ d.o.f. and imagine a

scenario where three space-like

separated observers A, B, C have access to the first, second and third spin- $1/2$ respectively:



When one of the observers measure $\hat{X}$ and the other two measure $\hat{Y}$, then the product of the three measurements is always $+1$

because $|\psi\rangle$ is an eigenstate of $\hat{X}_1 \hat{Y}_2 \hat{Y}_3$, $\hat{Y}_1 \hat{X}_2 \hat{Y}_3$, and $\hat{Y}_1 \hat{Y}_2 \hat{X}_3$

with eigenvalue $+1$. Let's denote measurement outcomes by lower-case letters, thus, $x_1, y_2, y_3 = \pm 1$ etc.

If one now uses the classical intuition that the products of several constants of motion is also a constant of motion, then one concludes that

$$\begin{aligned} x_1 x_2 x_3 &= x_1 y_2 y_3 \cdot y_1 x_2 y_3 \cdot y_1 y_2 x_3 \\ &= 1 \times 1 \times 1 = 1 \end{aligned}$$

Therefore, classical intuition implies that if all three observers (A, B, C) choose to measure their spins X operator, then the product of measurement outcomes will be $+1$. However,

$$x_1 \otimes x_2 \otimes x_3 |\psi\rangle = -|\psi\rangle \quad !$$

$\Rightarrow$ the classical intuition is incompatible with QM.