Basic structure and postulates of QM.

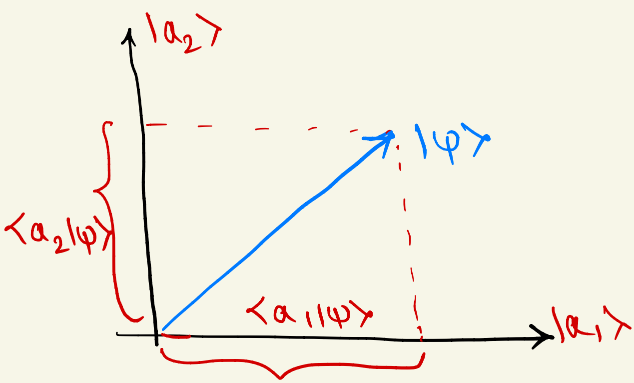
(Read Shankar Ch.1 or Gottfried Ch.2)

- The states, denoted by the symbol $|\varphi\rangle$, (ket $|\varphi\rangle$) live in a Hilbert space (a vector space equipped with a scalar product).

- One may chose a basis of the Hilbert space, e.g. $\{|a\rangle\}$ and all states can be expanded in a chosen basis:

$$
\begin{aligned}
|\varphi\rangle &= \underbrace{\mathbb{1}}_{\text{identity operator}}|\varphi\rangle \\
&= \sum_a |a\rangle\langle a|\varphi\rangle \\
&= \sum_a \underbrace{\langle a|\varphi\rangle}_{\text{a complex number.}} |a\rangle
\end{aligned}
$$

For example, if the basis is two-dimensional, then pictorially:



The basis used to express a state $|\psi\rangle$ is a <u>choice</u>. One may as well use a different basis. The components in two different basies are related by a <u>unitary</u> transformation

$$|\psi\rangle = \sum_a \langle a|\psi\rangle\, |a\rangle$$
$$= \sum_\alpha \langle \alpha|\psi\rangle\, |\alpha\rangle$$

$$\Rightarrow \quad \underbrace{\langle\alpha|\psi\rangle}_{\psi_\alpha} = \sum_a \underbrace{\langle a|\psi\rangle}_{\psi_a}\, \underbrace{\langle\alpha|a\rangle}_{U_{\alpha a}}$$

$$\Rightarrow \quad \psi_\alpha = \sum_a U_{\alpha a}\, \psi_a$$

U is a unitary matrix i.e. $U^\dagger U = \mathbb{1}$

Check: $$(U^\dagger U)_{ab} = \sum_\alpha (U^\dagger)_{a\alpha}\, U_{\alpha b}$$
$$= \sum_\alpha U^*_{\alpha a}\, U_{\alpha b} = \sum_\alpha \langle a|\alpha\rangle\langle\alpha|b\rangle$$
$$= \langle a|b\rangle = \delta_{ab}.$$

The operators in QM act on states to yield states.

$$\hat{O} \, |\psi_1\rangle = |\psi_2\rangle$$

If one chooses a basis for the Hilbert space, the above equation can be thought as action of a matrix on a vector to yield another vector:

$$\langle a | \hat{O} \, |\psi_1\rangle = \langle a | \psi_2\rangle$$

$$\Rightarrow \sum_b \langle a | \hat{O} \, | b\rangle \langle b | \psi_1\rangle = \langle a | \psi_2\rangle$$

$$\Rightarrow \sum_b \hat{O}_{ab} \, \psi_{1b} = \psi_{2a}$$

Therefore, states ~ vector, operators ~ matrices
One can multiply and add operators to obtain other operators.

One can add states to obtain other states, but there is no meaningful multiplication of states within the same Hilbert space.
[states live in a vector space, operators in a ring].

There are just two postulates :

1. The information about a system is encoded in its wave-f^n $|\psi(t)\rangle$ (in an appropriate Hilbert space) that evolves according to the Schrodinger's eqn :

$$i \frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(t)\rangle$$

[fully deterministic]

where $\hat{H}$ is a hermitian operator called Hamiltonian. Note that this equation implies that $\langle\psi(t)|\psi(t)\rangle = 1 \;\; \forall\, t$.

2. If a measurement of an observable $\hat{\Omega}$ is made on the system, then the measurement will always result in an outcome ω that is one of the eigenvalues of $\hat{\Omega}$.

The outcome is intrinsically probabilistic, with the probability for outcome ω, $p(\omega) = |\langle\omega|\psi(t)\rangle|^2$. Right after the measurement, the wave-f^n is given by ("wavef^n collapse"):

$$|\psi'\rangle = \frac{\langle\omega|\psi\rangle\ |\omega\rangle}{\sqrt{|\langle\omega|\psi\rangle|^2}}.$$

One can only measure commuting operators simultaneously.

<u>More details</u>:

a) The postulate #1 describes systems with <u>continuous time</u> evolution. One can also consider a case where wavefunction abruptly changes from $|\psi_0\rangle$ to $U|\psi_0\rangle$ at some point in time, where U is a unitary operator i.e. $U^\dagger U = \mathbb{1}$. Physically, this may be thought of as a special case of #1 where the Hamiltonian $\hat{H}$ is non-zero only for an infinitesimal duration 'dt' during which it is infinitely large.

e.g. if $\hat{H}(t) = \hat{h}\ \delta(t - t_0)$ where $\delta(x)$ is the Dirac-delta fn and $\hat{h}$ is some operator, then the Schrodinger's eqn implies,

$$|\psi(t_0 + dt)\rangle =$$

$$|\psi(t_0 - dt)\rangle - i\,\hat{h} \int_{t_0 - dt}^{t_0 + dt} \delta(t - t_0)\ \hat{h}\ dt\ |\psi(t)\rangle$$

$$\Rightarrow |\psi(t_0 + dt)\rangle \simeq \hat{U}\ |\psi(t_0 - dt)\rangle$$

where $\hat{U} = \mathbb{1} - i\,\hat{h}$.

(b) If the eigenspectrum of the measured operator $\hat{\Omega}$ has degeneracies, then postulate #2 as stated above is ambiguous. This is easily fixed by modifying it as:

$$|\psi'\rangle = \frac{P_\omega |\psi\rangle}{\sqrt{\langle P_\omega \psi | P_\omega \psi\rangle}}$$

where P_ω is the projector onto the subspace of $\hat{\Omega}$ with eigenvalue ω.

For example, if there are two eigenvectors $|\omega_1\rangle$ and $|\omega_2\rangle$ of $\hat{\Omega}$, both with the same eigenvalue ω, then after measuring $\hat{\Omega}$, the probability of obtaining outcome ω is:

$$p(\omega) = |\langle\omega_1|\psi(t)\rangle|^2 + |\langle\omega_2|\psi(t)\rangle|^2,$$

and the state right after the measurement is given by,

$$|\psi'\rangle = N\underbrace{\left(|\omega_1\rangle\langle\omega_1| + |\omega_2\rangle\langle\omega_2|\right)}_{P_\omega}|\psi(t)\rangle$$

where N is the normalization $= \dfrac{1}{\sqrt{\langle\psi'|\psi'\rangle}}$.

(c) The eigenvectors of $\hat{H}$ play an important role in a large class of problems. This is because if a system is prepared in an eigenstate $|E\rangle$ of $\hat{H}$, then the time evolution is trivial:

$|\psi(t)\rangle = e^{-iEt}|E\rangle$, i.e. up to a phase factor (that is not relevant here), the state is simply $|E\rangle$ for all times. The eigenvalue problem for $\hat{H}$, namely,

$\hat{H}|E\rangle = E|E\rangle$ is called <u>time-independent Schrodinger Eqn.</u>

(d) Expectation value of an operator.

Consider several copies of the same state $|\psi\rangle$ and let's measure the same operator $\hat{\Omega}$ for each copy.

As mentioned above, the outcome is probabilistic. Therefore, one may ask about the <u>average</u> over the outcomes as number of copies $\to$ infinity. This is called the expectation value of $\hat{\Omega}$ w.r.t. $|\psi\rangle$.

If the eigenspectrum of $\hat{\Omega}$ is denoted as $|\omega\rangle$ (same notation as above), then,

$$\text{prob.}(\omega) = |\langle\omega|\psi\rangle|^2$$

Expectation value of $\hat{\Omega} \equiv \langle \hat{\Omega} \rangle$

$$= \sum_{\omega} \text{prob.}(\omega)\ \omega$$

$$= \sum_{\omega} |\langle \omega | \psi \rangle|^2 \omega$$

$$= \sum_{\omega} \omega \ \langle \psi | \omega \rangle \langle \omega | \psi \rangle$$

$$= \langle \psi | \Big(\sum_{\omega} \omega \, |\omega\rangle\langle\omega| \Big) \psi \rangle$$

$$= \langle \psi | \hat{\Omega} | \psi \rangle$$

where we have used $\hat{\Omega} = \sum_{\omega} \omega \, |\omega\rangle\langle\omega|$ (spectral decomposition of $\hat{\Omega}$).

Its worth emphasizing that this is just the average outcome obtained over a very large number of repeated measurements made on identical copies of $|\psi\rangle$.