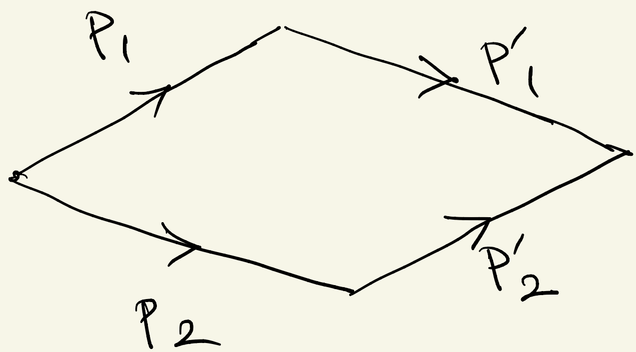
Landau's effective energy functional for a Fermi liquid.

One important consequence of above discussion about scattering near the Fermi surface is that the scattering is always in the forward direction

that is $\quad p_1 , \; p_2 \rightarrow p'_1 , \; p'_2$

where the set $\{p_1, p_2\} \simeq \{p'_1, p'_2\}$



Therefore to the leading approximation, the interaction between q.p.'s looks like

$$H_I \simeq c^{+}_{k_1} c^{+}_{k_2} c_{k_1} c_{k_2}$$

$$= n_{k_1} n_{k_2}.$$

Following such considerations, consider a fermi liquid with a given distribution of quasiparticles, which we label as n_p. The excitations of this system are labeled by δn_p, defined as

$$\delta n_{p\sigma} = n_{p\sigma} - n_{p\sigma}(T=0).$$

It's important to note that $n_p(T=0)$ is just a reference distribution, and only $\delta n_{p\sigma}$ is physically meaningful. δn_p can be defined both for equilibrium settings, as well as out-of-equilibrium problems. Let us first consider the system at equibrium at $T \neq 0$.

To determine $n_{p\sigma}$ at $T \neq 0$, we will minimize the free energy of the system $F(T) = E - TS$ w.r.t. δn.

To do so, we need an expression for E and an expression for S.

Since quasiparticles are fermions, with eigenvalues for the number operator $\hat{n}_{p\sigma} = 0, 1$, the expression for entropy is simply given by the Shannon entropy of the distribution $\{n_{p\sigma}\}$

$$S = -\sum_{p\sigma} \left[n_{p\sigma} \log n_{p\sigma} + (1-n_{p\sigma}) \log(1-n_{p\sigma}) \right]$$

The expression for energy $E(\{n_{p\sigma}\})$ is more interesting and is the essence of Landau fermi liquid theory. If q.p.'s were non-interacting, then one expects that $E = \sum_{p\sigma} (\varepsilon^0_{p\sigma} - \mu) n_{p\sigma}$ where $\varepsilon^0_{p\sigma} = \frac{p^2}{2m}$ where m is the electron mass. However, since q.p.'s do interact, perhaps one can simply replace $m \to m^*$, the effective mass. This, however, would be incorrect.

At low T, $\delta n_{p\sigma}$ is non-zero only in a small window δ ($\approx \frac{T}{E_F}$ if the system is at equilibrium).

Therefore the expression $\sum_p (\varepsilon^0_{p\sigma} - \mu)\delta n_{p\sigma}$ is order δ^2 since $\varepsilon_{p\sigma} - \mu$ is of order δ in this small window. For example, recall the calculation for specific heat C at $T \ll E_F$, where for non-interacting e^-s $\delta \sim T/E_F$, leading to $\delta E = E(T) - E(T=0) \sim T^2/E_F \Rightarrow C \sim \frac{T}{E_F}$.

To be consistent, we need to keep all terms of order δ^2. Since q.p.s interact, one can write down the following additional contribution to the energy:

$$\frac{1}{2} \sum_{\substack{pp' \\ \sigma\sigma'}} \delta n_{p\sigma}\, \delta n_{p'\sigma'}\, f_{\substack{pp' \\ \sigma\sigma'}}$$

where $f_{\substack{pp' \\ \sigma\sigma'}}$ are some numbers. This term is also of order δ^2.

$f_{pp'}$ can be thought of as the interaction energy between quasiparticle excitations with momenta p and p'. [Note that, some books/papers put a factor of $\frac{1}{V}$ in front of every $f_{pp'}$, e.g. $\frac{1}{2V^2} \sum_{pp'} \delta n_p \delta n_{p'} f_{pp'}$, this changes the overall scale and dimension of $f_{pp'}$ and is simply a matter of convention. One only needs to make sure that the energy E is extensive, whatever the convention].

Combining everything, we now have an expression for F upto $O(\delta^2)$ where δ denotes the density of quasiparticles (i.e. we are doing a low-density expansion):

$$F = E - TS =$$

$$= \sum_{p\sigma} \varepsilon^0_{p\sigma} n_{p\sigma} + \frac{1}{2} \sum_{\substack{p\sigma \\ p'\sigma'}} f_{pp' \atop \sigma\sigma'} \delta n_{p\sigma} \delta n_{p'\sigma'}$$

$$+ T \sum_{p\sigma} \left[n_{p\sigma} \log n_{p\sigma} + (1-n_{p\sigma}) \log (1-n_{p\sigma}) \right]$$

where $\varepsilon_{p\sigma} = \dfrac{p^2}{2m^*}$ (m^* = effective mass),

$$\delta n_{p\sigma} = n_{p\sigma}(T) - n_{p\sigma}(T=0)$$

and $n_{p\sigma}(T)$ is as yet undetermined.

To determine $n_{p\sigma}(T)$, we now minimize F w.r.t. $n_{p\sigma}$.

$$\frac{\delta F}{\delta n_{p\sigma}} = \frac{\delta E}{\delta n_{p\sigma}} - T \frac{\delta S}{\delta n_{p\sigma}} = 0$$

One can call $\frac{\delta E}{\delta n_{p\sigma}}$ as the energy of the quasiparticle with momentum, spin $= p, \sigma$, in analogy with non-interacting electrons where it simply equals $\frac{p^2}{2m}$. Here, we find

$$\varepsilon_{p\sigma} = \varepsilon^0_{p\sigma} + \sum_{p'\sigma'} f_{p\sigma, p'\sigma'} \, \delta n_{p'\sigma'}$$

i.e. the energy of a quasiparticle depends on the full distribution $\{\delta n\}$, i.e. it depends on the presence/absence of all excitations. Similarly, one finds $\frac{\delta S}{\delta n_{p\sigma}} = -\log\left(\frac{n_{p\sigma}}{1-n_{p\sigma}}\right)$.

In contrast to $\frac{\delta E}{\delta n_{p\sigma}}$, this only depends on $n_{p\sigma}$ and not full $\{\delta n\}$.

Equating $\frac{\delta F}{\delta n_{p\sigma}} = \frac{\delta E}{\delta n_{p\sigma}} - T \frac{\delta S}{\delta n_{p\sigma}} = 0$

$$\Rightarrow \quad n_{p\sigma} = \frac{1}{e^{\beta(\varepsilon_{p\sigma} - \mu)} + 1}$$

where $\varepsilon_{p\sigma} = \varepsilon^0_{p\sigma} + \sum_{p'\sigma'} f_{\substack{p\sigma \\ p'\sigma'}} \, \delta n_{p'\sigma'}$

$$= \frac{p^2}{2m^*} + \sum_{p'\sigma'} f_{\substack{p\sigma \\ p'\sigma'}} \left[n_{p'\sigma'}(T) - n_{p'\sigma'}(T=0) \right].$$

Unlike the case of non-interacting e^-s where $n^0_{p\sigma} = \frac{1}{e^{\beta(\varepsilon^0_{p\sigma} - \mu)} + 1}$,

here both sides of the equation depend on $n_{p\sigma}(T)$. Therefore, as such, it is a complicated integral equation. Furthermore, μ is also temperature dependent as usual.

The key to make progress would be to utilize the fact that at $T \ll \mu (\approx E_F)$, δn is small, and therefore we can <u>linearize</u> the above integral equation.

We also note that at $T=0$,

$\delta n = 0 \Rightarrow n_{p\sigma}(T=0) = \Theta(E_F - \varepsilon^0_{p\sigma})$.

where $E_F = \mu(T=0)$ is the fermi energy, and Θ is the Heaviside f^n.

Assuming rotational symmetry, the Fermi energy remains unchanged compared to the non-interacting e^-s due to Luttinger theorem.

Linearized equation for $n_p(T)$:

At temperatures $T \ll \mu$, one may linearize the above equation for $n_{p\sigma}(T)$:

$$n_{p\sigma}(T) = \frac{1}{e^{\beta(\varepsilon^0_{p\sigma} - \mu)} + 1} + \left.\frac{\partial n_F(x)}{\partial x}\right|_{x=\beta(\varepsilon^0_p - \mu)} \sum_{p'} \beta\, \delta n_{p'\sigma'} f_{pp' \sigma\sigma'}$$

where $n_F(x) = \dfrac{1}{e^x + 1}$ is the Fermi f^n.

Since $T \ll \mu$, $n_F(x)$ is almost a Delta f^n with width T/T_F. Therefore, to the leading order, one may assume that $f_{pp'}$ depends only on the unit vectors $\hat{p}, \hat{p}'$ and not the magnitudes p, p'.

Further, by rotational symmetry,

$\delta n_{p\sigma}$ only depends on the magnitude p and not $\hat{p}$. Therefore the sum

$$\sum_{p'\sigma'} \delta n_{p'\sigma'} f_{\substack{pp' \\ \sigma\sigma'}}$$

$$\approx \int dp' \, \delta n_{p'\sigma'} \int d\Omega' \, f_{pp'}(\Omega')$$

where Ω' denotes angular integration.

Since the number of particles is conserved, $\int dp' \, \delta n_{p'\sigma'} = 0$.

Therefore, to the leading order, the distribution $n_{p\sigma}$ is infact just given by the standard Fermi function

$$n_{p\sigma} \simeq \frac{1}{e^{\beta(\varepsilon^0_{p\sigma} - \mu)} + 1}$$

where $\varepsilon^0_{p\sigma} = \dfrac{p^2}{2m^*}$.