# Collisionfree Collective Excitations (Zero sound)

Let us first recall the Boltzman equation for a distribution $g(k,r,t)$ [see eg Ashcroft-Mermin chapter 16].

$$\frac{\partial g}{\partial t} + \frac{\partial g}{\partial r}\cdot\frac{d\vec{r}}{dt} + \frac{\partial g}{\partial \vec{k}}\cdot\frac{\partial \vec{k}}{\partial t} = \left.\frac{\partial g}{\partial t}\right|_{\text{collision.}}$$

where $\left.\frac{\partial g}{\partial t}\right|_{\text{collision}}$ is the change in $g$ per unit time due to collisions and for nearly free fermions may be schematically written as

$$\left.\frac{\partial g}{\partial t}\right|_{\text{collision}} = -\int dk' \Big\{ \underbrace{W_{k,k'}\, g(k)[1-g(k')]}_{\text{particles leaving } k \text{ due to collisions}} - \underbrace{W_{k',k}\, g(k')[1-g(k)]}_{\text{particles entering } k \text{ due to collisions.}} \Big\}$$

where $W_{k,k'}$ is the scattering probability from $k \to k'$.

Also, recall that within the relaxation time approximation, the collision term is given by

$$\left.\frac{\partial g}{\partial t}\right|_{\text{collision}} = -\frac{[g(\vec{k}) - g^0(\vec{k})]}{\tau(\vec{k})}$$

where $\tau(\vec{k})$ is the scattering lifetime.

The above equation makes sense only semi-classically since we are specifying both the momenta and position simultaneously. We would like to write down a similar equation for the quasiparticle distribution that is varying in $\vec{r}, \vec{k}, t$.

## Two regimes:

In a Fermi-liquid, as discussed earlier, the quasiparticle scattering rate takes the form $\Gamma \sim \tau^{-1} \sim (\varepsilon_k - \varepsilon_F)^2 + T^2$.

Consider a perturbation of the system which leads to the q.p. distribution $n_p(\vec{r},t)$ varying as:

$$n_p(\vec{r},t) = n_p^0 + \delta n_p(\vec{q},\omega)\, e^{i(\vec{q}\cdot\vec{r} - \omega t)}$$

If the frequency $\omega \gg \frac{1}{\tau}$, then the effect of collisions may be neglected, since many oscillation cycles have been completed before two q.p.'s collide.

In this limit, as we will soon see, the system undergoes dissipation-free oscillations as long as the velocity of the corresponding wave is larger than the Fermi velocity (so that the mode is not damped by the continuum of particle-hole excitations). This mode is called 'zero sound'.

In the other limit, $\omega\tau \ll 1$, collisions can't be neglected and one obtains a sound mode that is similar to the ordinary sound in liquids. This mode is often referred to as 'first sound'. Here we will primarily focus on the zero sound.

## Zero sound:

In the absence of collisions, the Boltzmann equ. for $n_p(r,t)$ is

$$\frac{\partial n_p}{\partial t} + \frac{\partial n_p}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial t} + \frac{\partial n_p}{\partial \vec{p}} \cdot \frac{\partial \vec{p}}{\partial t} = 0.$$

Using semiclassical dynamics,

$$\frac{d\vec{r}}{dt} = \vec{v} = \frac{\partial \varepsilon_p}{\partial \vec{p}}, \quad \frac{\partial \vec{p}}{\partial t} = -\frac{\partial \varepsilon_{\vec{p}}}{\partial \vec{r}}$$

where $\varepsilon_p = \underset{\left(=\frac{p^2}{2m^*}\right)}{\varepsilon^0_p} + \sum_{p'} f_p \, \delta n_{p'}$

and we have suppressed the spin-label (we will soon specialize to the case where oscillations are spin-symmetric).

Writing $n_p(\vec{r}, t) = n^0_p + \delta n_p(\vec{r}, t)$

where $n^0_p = \Theta\left(E_F - \frac{p^2}{2m^*}\right)$ is the $T=0$ equilibrium distribution, at the linear order in $\delta n_p$ one obtains,

$$\frac{\partial \, \delta n_p(\vec{r}, t)}{\partial t} + \frac{\partial \delta n_p(\vec{r}, t)}{\partial r} \cdot \frac{\partial \varepsilon_{\vec{p}}}{\partial \vec{p}} - \frac{\partial n^0_p}{\partial \vec{p}} \cdot \sum_{p'} f_{pp'} \frac{\partial \delta n_{p'}(\vec{r}, t)}{\partial \vec{r}} = 0 .$$

Assuming that these free oscillations occur at wavevector $\vec{q}$ and frequency $\omega$, we may write $\delta n_p(\vec{r}, t) = \delta n_{\vec{p}} \, e^{i q \cdot r - i\omega t}$

$$\Rightarrow (\vec{q}\cdot\vec{v}_p - \omega)\,\delta n_p - \vec{q}\cdot\vec{v}_p \frac{\partial n_0}{\partial \varepsilon_p} \sum_{p'} f_{pp'}\,\delta n_{p'} = 0$$

where we have used $\frac{\partial \varepsilon_p}{\partial \vec{p}} = \vec{v}_p$,

and $$\frac{\partial n^0_p}{\partial \vec{p}} = \frac{\partial n^0_p}{\partial \varepsilon_p}\frac{\partial \varepsilon_p}{\partial \vec{p}} = \frac{\partial n^0_p}{\partial \varepsilon_p}\vec{v}_p .$$

Let's choose coordinates so that $\vec{q} = q\hat{z}$ and $\vec{p} = p_F[\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta]$.

Further, let's parametrize $\delta n_p$ as

$$\delta n_p = \delta(\varepsilon_p - E_F)\, v_F\, u_p .$$ Since

$$\frac{\partial n_0}{\partial \varepsilon_p} = \delta(\varepsilon_p - E_F),$$ the factor of

$\delta(\varepsilon_p - E_F)$ drops out and we obtain,

$$(q v_F \cos\theta - \omega)\, u(\theta) - q v_F \cos\theta \int \frac{d\Omega}{4\pi} f(\theta-\theta')\, u(\theta') = 0$$

where $d\Omega = 2\pi \sin\theta' d\theta'$ is the differential solid angle.

$$(\cos\theta - \lambda)\, u(\theta) - \frac{\cos\theta}{2} \int f(\theta-\theta')\, u(\theta') \sin(\theta')\, d\theta' = 0$$

where $\lambda = \dfrac{\omega}{q v_F}$. This is an eigenvalue problem with matrix kernel $f(\theta-\theta')\sin(\theta')$

Although one may be able to solve it for specific f^n s $f(\theta-\theta')$, let's specialize to the case when only the Landau parameter $f^s_{l=0}$ is non-zero. In this case, one obtains,

$$(\cos\theta - \lambda)\, u(\theta) - \frac{\cos(\theta)}{2} F_0 \int u(\theta') \sin(\theta')\, d\theta' = 0$$

where $F_0 = f_{l=0}\, V\, \nu(E_F)$

Denoting $\int u(\theta') \sin(\theta')\, d\theta' = x$

$$\Rightarrow \quad u(\theta) = \frac{\cos(\theta)\, F_0\, x}{2(\cos\theta - \lambda)}$$

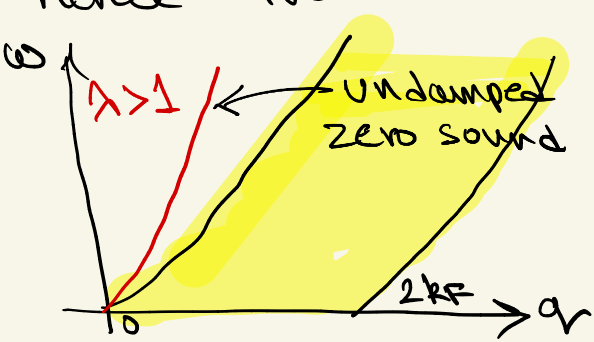
Using the definition of $\lambda$, this implies the following consistency condition (i.e. eigenvalue equation).

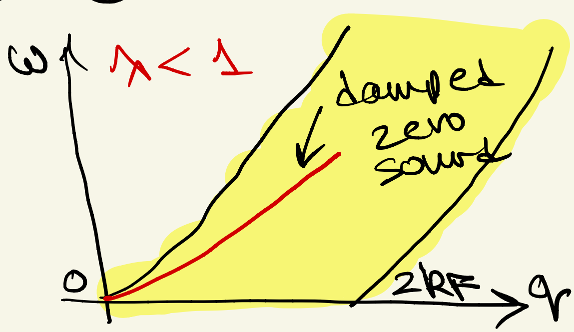
$$\frac{F_0}{2} \int \frac{\cos(\theta)\sin(\theta)\, d\theta}{[\cos(\theta) - \lambda]} = 1$$

$$\Rightarrow \frac{F_0}{2}\left[\lambda \log \frac{\lambda - 1}{\lambda + 1} + 2\right] = 1$$

$$\Rightarrow \boxed{\frac{\lambda}{2} \log \frac{\lambda+1}{\lambda-1} - 1 = \frac{1}{F_0}}$$

Recall $\lambda = \dfrac{\omega}{q v_F}$ and $\lambda > 1$ that the zero sound velocity $> v_F$ and hence the zero sound is undamped.



From above implicit equation for $\lambda$, one may verify that when $F_0 > 0$, there is one real solution and which has $\lambda > 1$. In He-3, $F_0 \approx 0.91 > 0$, and therefore zero-sound is undamped at small frequencies as also observed experimentally (see the papers posted on Canvas).

When $F_0 \gg 1$, $\lambda$ is also large and one may then solve the above equation by Taylor expansion,

$$\frac{\lambda}{2} \log\left[\frac{\lambda+1}{\lambda-1}\right] \simeq \frac{\lambda}{2}\left\{\log\left[1+\frac{1}{\lambda}\right] - \log\left[1-\frac{1}{\lambda}\right]\right\}$$

$$\simeq \frac{2\lambda}{2}\left[\frac{1}{\lambda} + \frac{1}{3\lambda^3} + \cdots\right] \simeq 1 + \frac{1}{F_0}$$

$$\Rightarrow \qquad \lambda \simeq \sqrt{\frac{F_0}{3}}\,.$$