Low-temperature Thermodynamics of a Fermi liquid.

(a) Specific heat :

To calculate the specific heat, we simply note that the leading contribution comes from states close to the Fermi surface with energy $|\varepsilon_p - \mu| \lesssim T$. As just discussed, at this order,

$$n_p(T) \simeq \frac{1}{e^{\beta(\varepsilon^0_{p0} - \mu)} + 1}$$, and therefore

one may carry out the standard Sommerfeld expansion, leading to

$$\frac{C(V,T,N)}{V} = \frac{\pi^2}{2} \frac{T\, n}{\varepsilon_F}$$

where $n = \frac{N}{V}$ and $\varepsilon_F = \frac{p_F^2}{2m^*}$

n and p_F are related as usual:

$n = \frac{p_F^3}{3\pi^2}$. Note again that p_F is unchanged compared to non-interacting e⁻s (Luttinger theorem), and the only effect of interactions is that the electron mass is replaced by the effective mass m^*. Using above relations one may also write,

$$\frac{C(T,V,N)}{V} = a\, T\, n^{1/3}\, m^*$$

where a is a numerical constant.

Therefore, the specific heat is

enhanced by a factor of $\frac{m^*}{m}$ compared to the non-interacting case.

One may now calculate other quantities such as entropy and the temperature dependence of the chemical potential.

Entropy : $$T \frac{\partial s_a}{\partial T} = \frac{C}{V} = a T n^{1/3} m^*$$

where $s_a = \frac{S(T, V, N)}{V}$.

$$\Rightarrow \quad s_a = a n^{1/3} m^* T + \underbrace{f(N,V)}_{s_a(T=0)}$$

When there are no particles i.e. when when $N=0$, we expect that $s_a = 0$.

Further, $s_a(T=0) = 0$ from the third law of thermodynamics.

$$\Rightarrow \quad f(N,V) = 0 \Rightarrow s_a = \frac{C}{V} = a n^{1/3} m^* T$$

Chemical potential:

One may use the Maxwell relation,

$$\frac{\partial \mu}{\partial T} = -\frac{\partial S}{\partial N} = -\frac{\partial (S/V)}{\partial n}$$

$$= -a\left[\frac{1}{3} n^{-2/3} m^* T + n^{1/3} T \frac{\partial m^*}{\partial n}\right]$$

$$\Rightarrow \mu(T,n) = \underset{(=\varepsilon_F)}{\mu(T=0)} - \frac{aT^2}{2}\left[m^* n^{-2/3} + n^{1/3}\frac{\partial m^*}{\partial n}\right]$$

We note that in addition to the standard contribution proportional to $-T^2 n^{-2/3} m^*$, here one gets an additional term proportional to $\frac{\partial m^*}{\partial n}$. See Baym and Pethik for an alternative derivation of C, κ_s, μ.

Compressibility at $T = 0$

Compressibility K is given by $k = -\frac{\partial V}{V \partial P}$

$= \frac{1}{n^2} \frac{\partial n}{\partial \mu}$ where n is the density.

Using the integral equation for $n_{p\sigma}$,

$$\delta n_{p\sigma} = \frac{\partial n_{p\sigma}}{\partial \varepsilon_{p\sigma}} \left[\delta \varepsilon_{p\sigma} - \delta \mu \right]$$

Note that $\delta \varepsilon_{p\sigma} = \delta \sum_{p'\sigma'} f^{pp'}_{\sigma\sigma'} (n_{p'\sigma'} - n^0_{p'\sigma})$

depends implicitly on $\delta\mu$ due to the variation of $n_{p'\sigma}$ w.r.t. μ.

Summing over all $\vec{p}, \sigma$

$$\delta N = \sum_{p\sigma} \delta n_{p\sigma} = \sum_{p\sigma} \frac{\partial n_{p\sigma}}{\partial \varepsilon_{p\sigma}} \left[\sum_{p'\sigma'} f^{pp'}_{\sigma\sigma'} \delta n_{p'\sigma'} \right]$$

$$- \sum_{p\sigma} \frac{\partial n_{p\sigma}}{\partial \varepsilon_{p\sigma}} \delta \mu$$

As usual, the factor of $\frac{\partial n}{\partial \varepsilon}$ restricts both p and p' to lie at the Fermi surface $\left[\frac{\partial n}{\partial \varepsilon} \sim -\delta(\varepsilon - \varepsilon_F)\right]$

Let's consider the two terms on the right hand side separately.

$$-\sum_{p\sigma} \frac{\partial n_{p\sigma}}{\partial \varepsilon_{p\sigma}} \delta\mu = \int d\varepsilon\, \nu(\varepsilon)\, \delta(\varepsilon - \varepsilon_F)\, \delta\mu$$

$$= \nu(\varepsilon_F)\, \delta\mu$$

where $\nu(\varepsilon)$ denotes density of states (we use ν instead of N or n to make distinction with particle number or quasiparticle occupation).

The second term is $\sum_{p\sigma} \frac{\partial n_{p\sigma}}{\partial \varepsilon_{p\sigma}} \left[\sum_{p'\sigma'} f^{\sigma\sigma'}_{pp'}\, \delta n_{p'\sigma'}\right]$

$$\approx -\sum_{\substack{pp'\\ \sigma\sigma'}} f^{\sigma\sigma'}_{pp'}\, \delta n_{p'\sigma'}$$

$\in$ Fermi surface

The sum $\sum_{p\sigma} f_{pp'}_{\sigma\sigma'}$ can be simplied if $f_{pp'}_{\sigma\sigma'}$ is expanded in angular momentum modes in $d=2$, or using Legendre Polynomials in $d=3$. First, one utilizes the spin-rotational symmetry to write

$$f_{pp'}_{\sigma\sigma'} = f^{s}_{pp'} + f^{a}_{pp'}\,\sigma\sigma'$$

where f^s denotes the spin-symmetric and f^a the spin anti-symmetric component. f^s and f^a may then further be expanded in terms of angle θ between p and p':

$$d=3: \quad f^{s/a}_{pp'} = \sum_{\ell=0}^{\infty} f^{s/a}_{\ell} P_\ell(\cos\theta)$$

$d=2$: $f^{s/a}_{pp'} = \sum_{l=-\infty}^{\infty} f^{s/a}_{l}\, e^{il\theta}$

Returning to the sum $\sum_{p\sigma} f_{\substack{pp' \\ \sigma\sigma'}}$,

only the $l=0$ symmetric part survives $\sum_{p\sigma} f_{\substack{pp' \\ \sigma\sigma'}} = V f^{s}_{l=0}$

$\Rightarrow -\sum_{\substack{p\sigma \\ p'\sigma' \\ \in F.S.}} f_{\substack{pp' \\ \sigma\sigma'}} \delta n_{p'\sigma'} = -V f^{s}_{l=0} \delta N \; \nu(\varepsilon_F)$

where $\delta N = \sum_{p\sigma} \delta n_{p\sigma}$

Combining everything,

$$\delta N = \nu(\varepsilon_F)\left[\delta\mu - V f^{s}_{l=0}\,\delta N\right]$$

$$\Rightarrow \delta N = \frac{\delta\mu\ \nu(\varepsilon_F)}{1 + F^{s}_{l=0}}$$

where $F^{s}_{l=0} = V f^{s}_{l=0}\, \nu(\varepsilon_F)$

$\Rightarrow$ compressibility $\kappa = \frac{1}{n^2}\frac{\partial n}{\partial \mu}$

$$= \frac{1}{n^2}\frac{\nu(\varepsilon_F)/V}{1+F^s_{l=0}} .$$

Compared to the non-interacting fermions, one therefore obtains an additional factor of $1/(1+F^s_{l=0})$. Also, recall that the density of states $\nu(\varepsilon_F)$ is also enhanced by a factor of $\frac{m^*}{m}$ compared to the non-interacting $\bar{e}$'s.

(c) Spin susceptibility :

The spin-susceptibility is defined as $\chi = \frac{\partial m}{\partial h}$ where m is magnetization and h is the external magnetic field. The derivation for χ is almost identical to that for κ.

$\varepsilon_{p\sigma}$ in the presence of magnetic field h is given by

$$\varepsilon_{p\sigma} = \frac{p^2}{2m} - h\sigma n_{p\sigma} + \sum_{\sigma\sigma'} f_{pp'} \delta n_{p\sigma} \delta n_{p'\sigma'}$$

The variation of magnetization is

$$\delta m = \sum_{p\sigma} \sigma \delta n_{p\sigma} = \sum_{p\sigma} \frac{\partial n_{p\sigma}}{\partial \varepsilon_{p\sigma}} \sigma \Big[\sum_{p'\sigma'} f_{pp'} \delta n_{p'\sigma'} \Big]$$

$$- \sum_{p\sigma} \frac{\partial n_{p\sigma}}{\partial \varepsilon_{p\sigma}} \delta h$$

Again $p, p' \in$ F.S. because of the factor $\frac{\partial n}{\partial \varepsilon}$. The sum $\sum_{p\sigma} \sigma f^{a}_{pp'}_{\sigma\sigma'}$

$$= f^{a}_{l=0} V \sigma'$$

$$\Rightarrow \delta m = -\nu(\varepsilon_F) V f^{a}_{l=0} \delta m + \delta h \nu(\varepsilon_F)$$

$$\Rightarrow \frac{\delta m}{\delta h} = \frac{\nu(\varepsilon_F)}{1 + F^{a}_{l=0}}$$

Where $F^{a}_{l=0} = V f^{a}_{l=0} \nu(\varepsilon_F)$

Effective mass m^* :

Remarkably, one may deduce a relation between $\frac{m^*}{m}$ and Landau parameter $F_{l=1}^S$, if the system is Galilean invariant (e.g. a single-component He-3, but not a mixture of He-3 and He-4).

Lets denote the Hamiltonian in the lab frame as H, and consider a frame that is moving at velocity $\vec{u}$ w.r.t. the lab frame. The Hamiltonian H' and momentum $\vec{P}'$ in this frame are

$$H' = H - \vec{P}\cdot\vec{u} + \frac{1}{2}Mu^2, \quad \vec{P}' = \vec{P} - M\vec{u}$$

where $\vec{P}$ is the total momentum in the lab frame, and M is the total mass, which is frame invariant (because the system is Galilean invariant).

From now on, we will call the frame moving w.r.t. lab as the 'primed frame'.

Consider adding a single fermion with momentum $\vec{p}$ to the system. As soon as it is added, it will get dressed by other fermions in the system, and become a quasiparticle with momentum $\vec{p}$ and energy $\varepsilon_{\vec{p}} = \frac{\vec{p}^2}{2m^*} + \sum_{pp'} f_{pp'} \delta n_{p'}$

(we ignore the spin index in this discussion as it doesn't matter). The total mass of the system changes by m, the mass of the bare fermion (e.g. mass of a single He-3 atom).

$\Rightarrow$ ΔE (change in energy in the lab frame) $= \varepsilon_p$

$\Delta \vec{P}$ (change in momentum in the lab frame) $= \vec{p}$

$\Delta M = m$.

Using the above mentioned relation between relation between the two frames (i.e. the usual Galilean transformation laws):

$\Delta E'$ (change in energy in the primed frame)

$$= \Delta E - \vec{v}\cdot\Delta\vec{p} + \frac{1}{2}(\Delta M)u^2 = \varepsilon_{\vec{p}} - \vec{p}\cdot\vec{u} + \frac{1}{2}mv^2$$

$\Delta\vec{p}'$ (change in momentum in the primed frame)

$$= \Delta\vec{p} - (\Delta M)\vec{v} = \vec{p} - m\vec{v}$$

Therefore, in the primed frame, the energy-momentum dispersion for a q.p. is

$$\varepsilon'_{\vec{p}-m\vec{v}} = \varepsilon_{\vec{p}} - \vec{p}\cdot\vec{u} + \frac{1}{2}mu^2$$

or, equivalently,

$$\varepsilon'_{\vec{p}} = \varepsilon_{\vec{p}+m\vec{v}} - \vec{p}\cdot\vec{v} - \frac{1}{2}mv^2.$$

Lets expand the R.H.S. to $O(u)$:

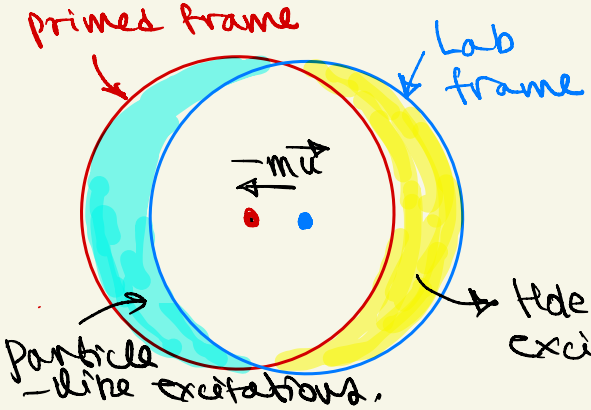
$$\varepsilon'_{\vec{p}} \approx \varepsilon_{\vec{p}} + \frac{\partial \varepsilon_{\vec{p}}}{\partial \vec{p}} \cdot m\vec{u} - \vec{p} \cdot \vec{u}$$

In the lab frame, there are no excitations, $\Rightarrow \delta n_p = 0 \Rightarrow \varepsilon_{\vec{p}} = \frac{p^2}{2m^*}$

$$\Rightarrow \varepsilon'_{\vec{p}} \approx \varepsilon_{\vec{p}} + m\frac{\vec{p}}{m^*} \cdot u - \vec{p} \cdot \vec{u}$$

$$= \varepsilon_{\vec{p}} + \left(\frac{m}{m^*} - 1\right)\vec{p} \cdot \vec{u}$$

On the other hand, in the primed frame, the distribution $n'_{\vec{p}}$ is a filled Fermi sea centered at $\vec{p}_0 = -m\vec{u}$.



Therefore, in this frame, there are <u>excitations</u> as shown in the figure on the left.

The energy cost of these excitations can be calculated using Landau's ansatz:

$$\varepsilon'_p = \varepsilon_p + \sum_{p'} f_{pp'} \, \delta n_{p'}$$

$$\delta n_{p'} = n_{p'+mu} - n_{p'}$$

$$= \frac{\partial n}{\partial \varepsilon_{p'}} \frac{\partial \varepsilon_{p'}}{\partial p'} \cdot m\vec{u} = \frac{\partial n}{\partial \varepsilon_{p'}} \frac{m \vec{p}' \cdot \vec{u}}{m^*}$$

$$\Rightarrow \varepsilon'_p = \varepsilon_p + \sum_{p'} \frac{\partial n}{\partial \varepsilon_{p'}} \frac{m \vec{p}' \cdot \vec{u}}{m^*} f_{pp'}$$

As usual, the factor of $\frac{\partial n}{\partial \varepsilon}$ restricts the sum to the Fermi surface. Since $\vec{p}'$ is a vector (i.e. angular momentum $= 1$), only $f^s_{\ell=1}$ will contribute. To find the exact prefactor, we use

$$\int d(\cos\theta) \, f^s_{\ell=1} P_{\ell=1}[\cos(\theta - \theta')] \cos(\theta')$$

$$= \int dx \, f^s_{\ell=1} \cos(\theta) \, x^2 \qquad (x = \cos\theta)$$

$$= \frac{1}{3} f^s_{\ell=1} \cos(\theta)$$

$$\Rightarrow \varepsilon'_p = \varepsilon_p - V \frac{f^s_{\ell=1}}{3} \mathcal{D}(E_F) \vec{p}\cdot\vec{u} \frac{m}{m^*}$$

$$= \varepsilon_p - \frac{F^s_{\ell=1}}{3} \vec{p}\cdot\vec{u} \frac{m}{m^*}$$

where $F^s_{\ell=1} = V f^s_{\ell=1} \mathcal{D}(E_F)$

Comparing with the alternate expression for ε'_p derived above,

$$\frac{m}{m^*} - 1 = -\frac{F^s_{\ell=1}}{3} \frac{m}{m^*}$$

$$\Rightarrow \frac{m}{m^*} = \frac{1}{1 + \frac{F^s_{\ell=1}}{3}}$$

$$\Rightarrow \frac{m^*}{m} = 1 + \frac{F^s_{\ell=1}}{3}$$