Integer Vs half-odd integer spinchains

There is a fundamental difference between integer spin ($S = 1, 2, 3, \ldots$) and half-odd integer spin ($S = 1/2, 3/2, 5/2, \ldots$) chains. In short, integer spin-chains can have a unique ground state with a finite energy gap while half-odd integer spin chains, such a possibility does not exist. There are analogous statements in higher dimensions as well.

We will discuss two different arguments that support the distinction between integer and half-odd integer spin chains:

(a) The original Lieb-Shultz-Mattis argument for $d=1$ (b) Oshikawa's argument for $d \geqslant 1$.

Lieb-Shultz-Mattis theorem for 1d Heisenberg AFM

Consider spin-1/2 Heisenberg chain with periodic boundary conditions: $H = J\sum_{i=1}^{L} S_i \cdot S_{i+1}$, $S_{L+1} = S_1$. Denoting the ground state as $|\varphi_0\rangle$, we will construct another low-lying (i.e. low energy) state $|\varphi_1\rangle$ with energy gap $E_1 - E_0 \lesssim \frac{1}{L}$. Then we will show that $\langle\varphi_1|\varphi_0\rangle = 0$ if spin S is half-odd integer. This implies that the spectrum for half-odd integer spin is gapless.

Given $|\varphi_0\rangle$, consider the following 'twisted' state $|\varphi_1\rangle$:

$$|\varphi_1\rangle = U|\varphi_0\rangle = e^{\frac{2\pi i}{L}\sum_{x=1}^{L} x\, S^z(x)}|\varphi_0\rangle$$

To show that $\langle\varphi_0|\varphi_1\rangle = 0$ for half-odd integer spins, let us consider their respective crystal momentum.

Let the crystal momentum of $|\varphi_0\rangle$ be k_0: $\quad T_x|\varphi_0\rangle = e^{ik_0}|\varphi_0\rangle$ where T_x is the lattice translation operator.

However, $T_x|\varphi_1\rangle = T_x U|\varphi_0\rangle$

$$= T_x U T_x^{\dagger} T_x |\varphi_0\rangle$$

Crucially, $T_x U T_x^{\dagger} = \pm U$

where $+$ sign corresponds to integer

spins and − corresponds to half-odd integer spins, as we now show:

$$T_x U T_x^+ = T_x e^{\frac{2\pi i}{L} \sum_{x=1}^{L} x S^z(x)} T_x^+$$

$$= e^{\frac{2\pi i}{L} \sum_{x=1}^{L} x \; S^z(x+1)}$$

$$= e^{\frac{2\pi i}{L} (S^z(2) + 2S^z(3) + \ldots + (L-1)S^z(L) + LS^z(1))}$$

$$= U e^{-\frac{2\pi i}{L} \sum_{x=1}^{L} S^z(x)} e^{2\pi i S^z(1)}$$

One can show that $S^z_{Total} = \sum_{x=1}^{L} S^z(x) = 0$ for the Heisenberg AFM spin-chain (and more generally, for an AFM on a bipartite lattice. i.e. $H = \sum_{(ij)} J_{ij} \vec{S}_i \cdot \vec{S}_j$ with $J_{ij} > 0$ and the lattice can be divided into two sublattices A, B so that $J_{ij} \neq 0$ only when $i \in A$ and $j \in B$). See, e.g. Auerbach sec. 5.1.

$\Rightarrow \quad T_x U T_x^{+} = e^{2\pi i s^z(1)} U$

When $S =$ integer, $s^z = -S, -S+1, \ldots, +S$ is also an integer, while when $S =$ half-odd integer, then s^z is also half-odd integer.

$\Rightarrow \quad T_x U T_x^{+} = \pm U$ where $+$ sign is for integer spin and $-$ sign for half-odd integer spin.

$\Rightarrow \quad T_x |\varphi_1\rangle = e^{i(k_0+\pi)} |\varphi_1\rangle$ for half-odd integer spin where k_0 is the ground state momentum.

$\Rightarrow |\varphi_1\rangle$ and $|\varphi_0\rangle$ differ in their lattice momentum by π $\Rightarrow$

$\langle \varphi_1 | \varphi_0 \rangle = 0$ for half-odd integer S.

Next, we show that $E_1 - E_0 \lesssim \frac{1}{L}$

where $E_1 = \langle \varphi_1 | H | \varphi_1 \rangle$, and $E_0 = \langle \varphi_0 | H | \varphi_0 \rangle$ is the ground state energy.

$$E_1 = \langle \varphi_1 | H | \varphi_1 \rangle$$

$$= \langle \varphi_0 | U^\dagger H U | \varphi_0 \rangle$$

Let's consider the action of U on a single bond:

$U^\dagger S^z(x) S^z(x+1) U = S^z_1 S^z_2$. since $[U, S^z(x)] = 0$

$U^\dagger S^+(x) S^-(x+1) U = U^\dagger S^+(x) U \, U^\dagger S^-(x+1) U$

Now, $e^{-i\alpha S^z} S^+ e^{i\alpha S^z} =$

$$S^+ - i\alpha [S^z, S^+] + \frac{(-i\alpha)^2}{2!} [S^z, [S^z, S^+]] + \dots$$

$$= S^+ - i\alpha S^+ + \frac{(-i\alpha)^2}{2!} S^+ + \dots$$

$= e^{-i\alpha} S^+$ (we are using $[S^z, S^+] = S^+$).

Taking h.c., $e^{-i\alpha S^z} S^- e^{i\alpha S^z} = e^{i\alpha} S^-$.

$$\Rightarrow \quad U^\dagger S^+(x) U = e^{-\frac{2\pi i}{L} x S^z(x)} S^+(x) e^{\frac{2\pi i}{L} x S^z(x)}$$

$$= e^{-\frac{2\pi i}{L} x} S^+(x)$$

and, $U^\dagger S^-(x+1) U = e^{\frac{2\pi i}{L}(x+1)} S^-(x+1)$

$$\Rightarrow U^\dagger S^+(x) S^-(x+1) U$$

$$= e^{\frac{2\pi i}{L}} S^+(x) S^-(x+1).$$

Similarly, $U^\dagger S^-(x) S^+(x+1) U$

$$\Rightarrow E_1 = \langle \varphi_0 | U^\dagger H U | \varphi_0 \rangle$$

$$= \langle \varphi_0 | U^\dagger \sum_x \Big[S^z(x) S^z(x+1) + \frac{1}{2} (S^+(x) S^-(x+1) + h.c. \Big] U | \varphi_0 \rangle$$

$$= \langle \varphi_0 | \sum_x S^z(x)\, S^z(x+1) | \varphi_0 \rangle$$

$$+ \frac{1}{2} e^{\frac{2\pi i}{L}} \langle \varphi_0 | \sum_x S^+(x)\, S^-(x+1) | \varphi_0 \rangle$$

$$+ \frac{1}{2} e^{\frac{-2\pi i}{L}} \langle \varphi_0 | \sum_x S^-(x)\, S^+(x+1) | \varphi_0 \rangle$$

$$= \langle \varphi_0 | \sum_x S^z(x)\, S^z(x+1) | \varphi_0 \rangle$$

$$+ \frac{1}{2} \cos\left(\frac{2\pi}{L}\right) \langle \varphi_0 | \sum_x S^+(x)\, S^-(x+1) + h.c. | \varphi_0 \rangle$$

$$+ \frac{1}{2} i \sin\left(\frac{2\pi}{L}\right) \langle \varphi_0 | \sum_x S^+(x)\, S^-(x+1) - h.c. | \varphi_0 \rangle$$

$$= E_0 + \frac{1}{2}\left[\cos\left(\frac{2\pi}{L}\right) - 1\right] \langle \varphi_0 | \sum_x S^+(x)\, S^-(x+1) + h.c. | \varphi_0 \rangle$$

$$+ \frac{1}{2} i \sin\left(\frac{2\pi}{L}\right) \langle \varphi_0 | \sum_x S^+(x)\, S^-(x+1) - h.c. | \varphi_0 \rangle$$

$\Rightarrow E_1 - E_0$

$$= \left[\cos\left(\frac{2\pi}{L}\right) - 1\right] \langle \varphi_0 | \sum_r \left[S^x(r) S^x(r+1) + S^y(r) S^y(r+1)\right] | \varphi_0 \rangle$$

$$+ \sin\left(\frac{2\pi}{L}\right) \langle \varphi_0 | \sum_r S^x(r) S^y(r+1) - S^y(r) S^x(r+1) | \varphi_0 \rangle$$

The term proportional to $\sin\left(\frac{2\pi}{L}\right)$ is zero because the Hamiltonian is invariant under the following anti-unitary transformation:

$S^x \to S^y$, $S^y \to S^x$, $S^z \to S^z$, $i \to -i$.

The term proportional to $\left[\cos\left(\frac{2\pi}{L}\right) - 1\right]$ is $O\left(\frac{1}{L}\right)$ since $\left[\cos\left(\frac{2\pi}{L}\right) - 1\right] \sim \frac{1}{L^2}$

and $\langle \varphi_0 | \sum_r S^x(r) S^x(r+1) + S^y(r) S^y(r+1) | \varphi_0 \rangle$

$\leq S^2 L \Rightarrow E_1 - E_0 \leq \frac{1}{L}$ Q.E.D

In general d, this yields

$$E_1 - E_0 \leq \frac{L^d}{L^2} \sim L^{d-2} \not\to 0 \text{ in } d \geq 2.$$

To appreciate Oshikawa's version of LSM theorem, which is more general than the original LSM argument discussed above, we first map the spin problem to the problem of interacting bosons using bosonic representation for spin-1/2.

$$S^+ = S^x + i S^y = b^+$$

$$S^- = b$$

$$S^z = b^+ b - \frac{1}{2}.$$

Since $-\frac{1}{2} \leq S^z \leq \frac{1}{2} \Rightarrow 0 \leq b^+ b \leq 1.$

When $\sum_x S^z(x) = 0 \Rightarrow \dfrac{\sum_{x=1}^{L} b^+(x)\, b(x)}{L} = \dfrac{1}{2}$

$\Rightarrow$ Bosons at half-filling.

Oshikawa's result is that for a system at filling $\nu = \frac{p}{q}$, system has atleast q low-lying states.

Unlike LSM argument, which works only in $d=1$. Oshikawa's argument works also in $d>1$.

Consider a system with the Hamiltonian

$$H = H_{hopping} + H_{interaction}$$

where $H_{hopping} = -t\,(b^{+}_{i}\, b_{j} + h.c.)$

$$H_{interaction} = V(\{n\}) \quad \text{where}$$

$V(\{n\})$ is some function of boson number operator e.g. $V = J \sum_{\langle rr' \rangle} n_r n_{r'}$.

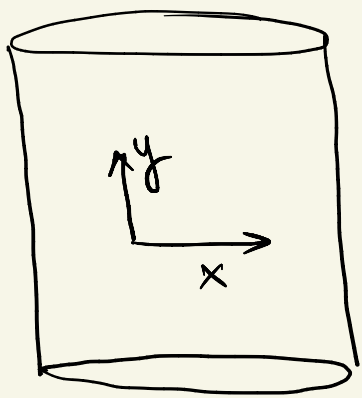
H is translationally invariant, this would be crucial for the argument.

Note that the spin-1/2 Heisenberg model is a class of such Hamiltonians with the constraint $0 \le n \le 1$.

$$S^+(r)\, S^-(r') + h.c. \longrightarrow b^+(r)\, b(r') + h.c.$$

$$S^z(r)\, S^z(r') \longrightarrow \left(n(r) - \tfrac{1}{2}\right)\left(n(r') - \tfrac{1}{2}\right).$$

Lets specialize to $d \le 2$ with system size $L_x \times L_y$ and impose periodic boundary conditions along the x-direction:



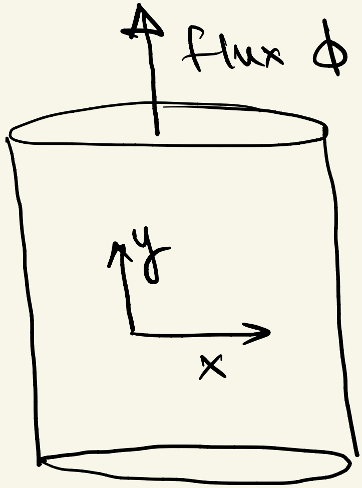
The argument is as follows:

Lets minimally couple the system to a gauge field where the corresponding conserved charge is just the total boson number.

$$b^\dagger_r b_{r'} + h.c.$$

$$\longrightarrow b^\dagger_r b_{r'} e^{i a_{rr'}} + h.c.$$

lets chose the gauge field so that it corresponds to a uniform magnetic field of flux $= \Phi$ through the cylinder.



One may chose the gauge field $a_{r,r+\hat{y}} = 0$, $a_{r,r+\hat{x}} = \frac{\phi}{L_x}$ to insert such a flux.

The Hamiltonian in the presence of the gauge field is

$$H(\phi) = -t \sum_{\vec{r}} \left(b^\dagger_r \, b_{r+\hat{x}} \, e^{\frac{i\phi}{L_x}} + h.c. \right)$$

$$- t \sum_{\vec{r}} \left(b^\dagger_r \, b_{r+\hat{y}} + h.c. \right)$$

$$+ V(\{n_r\}).$$

Lets adiabatically increase the flux ϕ from 0 to 2π.

Interestingly, $H(\phi=2\pi)$

same spectrum as $H(\phi=0)$. This is because they are unitarily related as follows

$$U \, H(\phi=2\pi) \, U^{\dagger} = H(\phi=0)$$

where $U = e^{\frac{2\pi i}{L_x} \sum_{\vec{r}} x \, n_{\vec{r}}}$. Note that U has the same form as the unitary in the original LSM argument. If the ground state of $H(\phi=0)$ is $|\psi_0\rangle$, it will evolve under this adiabatic into some state $|\psi'_0\rangle$ which would generically be different than $|\psi_0\rangle$.

The only constraint is that both $|\varphi_0\rangle$ and $|\varphi'_0\rangle$ have the same crystal momentum. This is because at each point during the time-evolution, $[T_x, H(\phi)] = 0$ $\Rightarrow$ T_x eigenvalue is a constant of motion. Thus,

$$T_x|\varphi_0\rangle = e^{ip_0}|\varphi_0\rangle \,, \quad T_x|\varphi'_0\rangle = e^{ip_0}|\varphi'_0\rangle$$

Now, let's consider the state $U|\varphi'_0\rangle$. By construction, it is a low-lying state:

$$\langle\varphi'_0| U^\dagger H(\phi=0)\, U|\varphi'_0\rangle$$
$$= \langle\varphi'_0| H(\phi=2\pi)|\varphi'_0\rangle \simeq E_0$$

since the adiabatic transformation

can take a ground state only to itself or a nearly degenerate state. The crystal momentum of the state $U|\psi_0'\rangle$ is, however, not necessarily same as that of the ground state $|\psi_0\rangle$:

$$T_x\, U\,|\psi'_0\rangle$$

$$= T_x\, U\, T_x^{\dagger}\, \underbrace{T_x|\psi'_0\rangle}_{= e^{iP_0}|\psi'_0\rangle}$$

Using the same calculation we did for the LSM theorem,

$$T_x U T_x^{\dagger} = U\, e^{2\pi i \frac{\sum n_r}{L_x}}$$

$$= U\, e^{2\pi i \nu L_y}$$

Writing $\nu = \frac{p}{q}$, lets chose L_y and q to be co-prime. then the crystal momentum of $U|\psi'_0\rangle$ differs from that of $|\psi_0\rangle$.

Repeating the argument q-times, one obtains q states, all of whom have low energy and which have distinct crystal momentum.

Q.E.D.

The spin-$1/2$ Heisenberg chain corresponds to $q=2$, and one obtains atleast one low-lying excited state, in agreement with LSM.