

Spin systems: large-S and large-N approaches.

Large-S expansion:

We employ the Holstein-Primakov (H-P) representation of bosons:

$$S_x + i S_y = S^+ = \sqrt{2S - n_b} b$$
$$S^z = S - n_b$$

where $n_b = b^\dagger b$ and one restricts

$0 \leq n_b \leq S$. This representation is useful when the spontaneous breaking of the spin-rotation $SU(2)$ symmetry is assumed to occur along z -axis with $\langle S^z \rangle \simeq S$ i.e. $n_b/S \ll 1$. Therefore this is a large-S (semiclassical) expansion. Let's consider ferro and anti-ferro cases.

Ferro magnet using HP bosons,

$$H = -|J| \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

Since in the ssB phase, all spins point along the same direction, the details of lattice (e.g. bipartite vs not) do not matter. Using HP bosons,

$$\begin{aligned} H &= -|J| \sum_{\langle ij \rangle} \left[S_i^z S_j^z + \frac{1}{2} (S_i^+ S_j^- + \text{h.c.}) \right] \\ &= -|J| \sum_{\langle ij \rangle} \left[(S - b_i^\dagger b_i) (S - b_j^\dagger b_j) \right. \\ &\quad \left. + \frac{1}{2} \left(\sqrt{2S - b_i^\dagger b_i} b_i b_j^\dagger \sqrt{2S - b_j^\dagger b_j} + \text{h.c.} \right) \right] \end{aligned}$$

Expanding $\sqrt{2S - b^\dagger b}$ in the limit $S \gg b^\dagger b$
or $\sqrt{2S} \left[1 - \frac{b^\dagger b}{2S} \right]^{1/2} \simeq \sqrt{2S} \left[1 - \frac{b^\dagger b}{4S} + \dots \right]$

and keeping the terms upto $O(s^0)$

in H , one finds,

$$H = H_0 + H_1 + H_2$$

$$\text{where } H_0 = -S^2 |J| \frac{N Z}{2} = O(s^2)$$

$$\begin{aligned} H_1 &= S |J| \left[Z \sum_i b_i^\dagger b_i - \sum_{\langle ij \rangle} (b_i^\dagger b_j + \text{h.c.}) \right] \\ &= \frac{S |J| Z}{N} \sum_k b_k^\dagger b_k \left(1 - \frac{1}{Z} \sum_{\vec{a}_r} e^{i \vec{k} \cdot \vec{a}_r} \right) = O(s). \end{aligned}$$

where Z is the coordination number

and the sum over $\vec{a}_r$ in H_1 corresponds

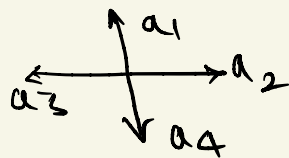
to sum over nearest neighbor bonds

of any given lattice site. e.g., on

the square lattice, $\frac{1}{Z} \sum_{\vec{a}_r} e^{i \vec{k} \cdot \vec{a}_r} \equiv \gamma_k$

$$= \frac{1}{4} (e^{i k_x} + e^{i k_y} + e^{-i k_x} + e^{-i k_y})$$

$$= \frac{\cos(k_x) + \cos(k_y)}{2}$$



Finally, H_2 is the interaction term between the bosons at $O(1)$ [there are higher order interaction terms and non-interaction terms as well].

$$H_2 = \frac{|J|}{4} \sum_{\langle i,j \rangle} [b_i^\dagger b_j^\dagger (b_i - b_j)^2 + \text{h.c.}]$$

Retaining, only $H_0 + H_1$,

$$H \simeq -S^2 |J| \frac{Nz}{2} + \sum_k \omega_k b_k^\dagger b_k$$

$$\text{where } \omega_k = S|J|z \left(1 - \frac{1}{z} \sum_{\vec{a}_r} e^{i\vec{k} \cdot \vec{a}_r}\right)$$

$$\simeq c k^2 \text{ at small } k$$

$$\text{where } c \simeq S|J|z.$$

Therefore, the Goldstone modes of the SSB have dispersion $\omega_k \sim k^2$.

Exactly at $T=0$, these modes are unoccupied and therefore the ground state has no fluctuations. This is because the order parameter

$\sum_i \mathbf{S}_i^z$ commutes with the

Hamiltonian. Note that H_0 precisely corresponds to the ground state energy of a classical ferromagnet.

At $T>0$, the Goldstone modes would be occupied with Bose distribution

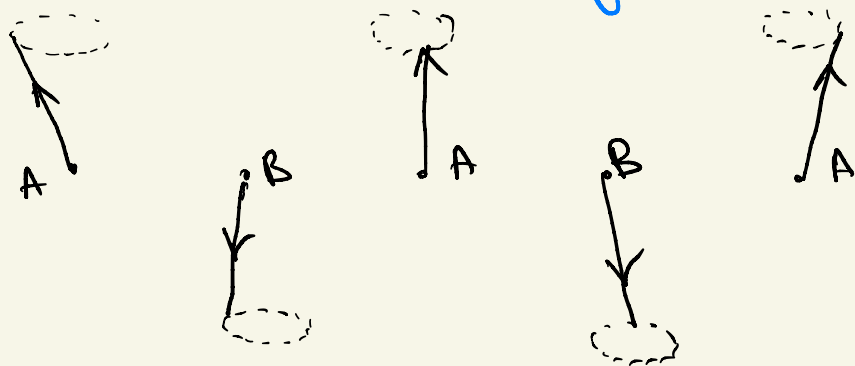
$n_B(k) = \frac{1}{e^{\beta \omega_k} - 1}$ and the question

of the stability of the ferromagnet is identical to the question of stability of a non-relativistic BEC at $T > 0$. The leading order correction to the ground state order parameter ($= S$) is given as

$$\begin{aligned}
 \langle S^z(T) \rangle &= \langle S^z(T=0) \rangle \\
 &= S - \langle b_i^\dagger b_i \rangle = S \\
 &= -\frac{1}{N} \sum_{\mathbf{k}} \langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rangle \\
 &= -\frac{1}{(2\pi)^d} \int_0^1 \frac{d^d k}{e^{\beta \epsilon_{\mathbf{k}}} - 1}
 \end{aligned}$$

The integral converges only in $d > 2 \Rightarrow$ ferromagnet is unstable against the thermal fluctuations in $d=1$ and $d=2$. In $d=2$, if instead of $SO(2)$ symmetric H , one had only a $U(1)$ symmetric H (e.g. due to an easy-plane anisotropy) then the system would allow for vortices leading to a finite T transition between a phase with power-law correlations for ferromagnetic order parameter, and a phase with exponentially decaying correlations (BKT transition).

Antiferromagnet using HP bosons



We restrict ourselves to a bipartite lattice. The spins point along $+\hat{z}$ direction the A sublattice and $-\hat{z}$ direction on the B sublattice.

Since the order parameter is

$(-1)^{x+y} \langle S(x,y) \rangle$, one can perform

a unitary transformation so that

H is expressed in terms of operators that vary smoothly in space.

On B sublattice, let's define

$$\tilde{S}^z = -S^z, \quad \tilde{S}^x = S^x, \quad \tilde{S}^y = -S^y$$

so that the commutation relations

$$[S^x, S^y] = i\epsilon_{xyz} S^z \text{ are obeyed by}$$

$\tilde{S}$ variables. Note that $S^+ = S^x + iS^y$

$$= \tilde{S}^x - i\tilde{S}^y = \tilde{S}^-. \text{ Therefore,}$$

$$H = |J| \sum_{\substack{\langle ij \rangle \\ i \in A \\ j \in B}} \left[S_i^z S_j^z + \frac{1}{2} (S_i^+ S_j^- + \text{h.c.}) \right]$$

$$= |J| \sum_{\langle ij \rangle} \left[-S_i^z \tilde{S}_j^z + \frac{1}{2} (S_i^+ \tilde{S}_j^+ + \text{h.c.}) \right]$$

Now we simply use the standard

HP bosons on both sublattices,

$$S^+ \text{ or } \tilde{S}^+ = \sqrt{2S - n_b} \, b$$

$$S^z \text{ or } \tilde{S}^z = S - n_b$$

Substituting the representation and again keeping the first few terms, one finds,

$$H = H_0 + H_1 + H_2 + \dots$$

$$\text{where } H_0 = -s^2 |J| \frac{Nz}{2}$$

$$H_1 = |J| s z \sum_k \left[b_k^\dagger b_k + \frac{\gamma_k}{2} (b_k^\dagger b_{-k}^\dagger + \text{h.c.}) \right]$$

$$\text{where } \gamma_k = \frac{1}{z} \sum_r e^{i\vec{k} \cdot \vec{a}_r} \text{ as before.}$$

$$\text{and } H_2 = -\frac{|J|}{4} \sum_{\langle ij \rangle} \left[b_i^\dagger (b_j^\dagger + b_i)^2 b_j + \text{h.c.} \right]$$

The crucial point to note is

that $b_k^\dagger / b_k$ do not diagonalize

H_1 due to the terms $(b_k^\dagger b_{-k}^\dagger + \text{h.c.})$.

This is a consequence of the fact that unlike a FM, where the classical ground state is the quantum ground state, for an AFM, the quantum (i.e. actual) ground state is not the product state $|\uparrow \downarrow \uparrow \dots\rangle$, but instead it contains quantum fluctuations.

This is because the order parameter $(-)^r \langle S(r) \rangle$, does not commute with H .

To diagonalize H , we employ the Bogolubov transformation, similar the case for a superfluid.

$$b_k = \cosh(\theta_k) \alpha_k + \sinh(\theta_k) \alpha_{-k}^\dagger$$

where α_k are standard boson operators, $[\alpha_k, \alpha_{k'}^\dagger] = \delta_{kk'}$ etc.

$$\Rightarrow H_1 = \frac{1}{2} \sum_k \left[\cosh(2\theta_k) \right.$$

$$+ \gamma_k \sinh(2\theta_k) \alpha_k^\dagger \alpha_k$$

$$+ \frac{1}{2} (\sinh(2\theta_k) + \gamma_k \cosh(2\theta_k))$$

$$(\alpha_k^\dagger \alpha_{-k}^\dagger + \text{h.c.})$$

$$+ \sinh^2 \theta_k + \frac{\gamma_k}{2} \sinh(2\theta_k) \Big]$$

One choose θ_k so that the terms $\alpha_k^\dagger \alpha_{-k}^\dagger$ vanish.

$$\Rightarrow \tanh(2\theta_k) = -\gamma_k.$$

Using this, H_1 becomes,

$$H_1 = \sum_k \omega_k \left(\alpha_k^\dagger \alpha_k + \frac{1}{2} \right) - \frac{J Z N S}{2}$$

where $\omega_k = |J| S Z \sqrt{1 - \gamma_k^2}$

ω_k vanishes at two k -points:

$\vec{k} = \vec{0}$ and $\vec{k} = (\pi, \pi, \dots, \pi)$ on
 $\equiv \vec{Q}$
 a cubic lattice in d -dimensions.

For $|k|$ small, $\gamma_k \simeq 1 - \frac{k^2}{Z}$

on a cubic lattice in d -dimensions
 with $Z = 2d$.

$$\Rightarrow \omega_k \simeq J S \sqrt{2Z} |k|$$

$$= c |k|.$$

Similarly, when $\vec{k} \approx \vec{Q}$,

$$\omega_k \approx J S \sqrt{2Z} |\vec{k} - \vec{Q}|$$

Let us calculate the ground state energy and order parameter.

Energy: In the ground state, the modes α_k are unoccupied.

$$\Rightarrow \alpha_k |0\rangle = 0$$

$$\begin{aligned} \Rightarrow \text{g.s. energy } E_{\text{g.s.}} &= E_{\text{cl}} \text{ (= classical energy)} \\ &+ \sum_k \frac{\omega_k}{2} - \frac{J S Z N}{2} \end{aligned}$$

$$= E_{\text{cl}} + \frac{1}{2} \sum_{\vec{k}} |J| S Z (\sqrt{1 - \gamma^2} - 1)$$

$$\text{where } E_{\text{cl}} = H_0 = - \frac{S^2 J N Z}{2}.$$

Since $\sqrt{1-y^2} < 1$

$$\Rightarrow E_{q.s.} < E_{cl}$$

$\Rightarrow$ Quantum fluctuations lower the energy compared to the classical ground state.

Order parameter:

Classically, at $T=0$, the order parameter is just S . Quantum fluctuations cause it to reduce by an amount

$$\Delta M = \frac{1}{N} \sum_{\vec{r}} e^{i\vec{Q} \cdot \vec{r}} \langle S_{\vec{r}}^z \rangle - S$$

$$= \frac{1}{N} \sum_{\vec{k}} [S - n_b(\vec{k})] - S$$

$$= - \int \frac{d^d k}{(2\pi)^d} \langle b_k^\dagger b_k \rangle$$

$$= \frac{1}{2} - \int \frac{d^d k}{(2\pi)^d} \frac{(n_B(k) + \frac{1}{2})}{\sqrt{1 - \gamma_k^2}}$$

The integral behaves as

$$- \int \frac{d^d k}{k (e^{\beta k} - 1)} + \frac{1}{2} \int \frac{d^d k}{k}$$

Therefore, it diverges at $T \gg 0$ in $d=1$, and at $T > 0$ in $d=2$. It is finite at $T > 0$ in $d \geq 3$.

Thus, quantum fluctuations destroy AFM ordering at

$T \geq 0$ in $d=1$ and at

$T > 0$ in $d=2$, as expected from Mermin-Wagner theorem.

One may summarize the results as,

	Ferromagnet	Antiferromagnet
$T = 0$	Stable in any d . $[M, H] = 0$. Classical g.s. = Quantum g.s.	Stable only in $d > 1$. $[M, H] \neq 0$. Classical g.s. $\neq$ Quantum g.s.
$T > 0$	Stable only in $d > 2$. $\omega_{\text{Goldstone}} \sim k^2$.	Stable only in $d > 2$. $\omega_{\text{Goldstone}} \sim k$.