Large-N Approach to spin-systems.

Shwinger bosons for SU(2) spins:

$$\vec{S} = b^\dagger \frac{\vec{\sigma}}{2} b \qquad b = [b_\uparrow \ b_\downarrow]$$

$$S^+ = b^\dagger_\uparrow b_\downarrow , \quad S^z = \frac{1}{2}(b^\dagger_\uparrow b_\uparrow - b^\dagger_\downarrow b_\downarrow).$$

with the constraint $b^\dagger_\uparrow b_\uparrow + b^\dagger_\downarrow b_\downarrow = 2S$

which ensures that $S^2 = S(S+1)$.

Check: $S_x^2 + S_y^2 + S_z^2 =$

$$\frac{1}{2}\Big[b^\dagger_\uparrow b_\downarrow b^\dagger_\downarrow b_\uparrow + b^\dagger_\downarrow b_\uparrow b^\dagger_\uparrow b_\downarrow$$

$$+ \frac{1}{4}(b^\dagger_\uparrow b_\uparrow b^\dagger_\uparrow b_\uparrow + b^\dagger_\downarrow b_\downarrow b^\dagger_\downarrow b_\downarrow$$

$$- 2 b^\dagger_\uparrow b_\uparrow b^\dagger_\downarrow b_\downarrow)$$

$$= \frac{1}{4}\Big[2n_\uparrow(1+n_\downarrow) + 2n_\downarrow(1+n_\downarrow)$$

$$+ n_\uparrow^2 + n_\downarrow^2 - 2n_\uparrow n_\downarrow \Big]$$

$$= \frac{1}{4}\Big[n_\uparrow^2 + n_\downarrow^2 + 2n_\uparrow n_\downarrow + 2(n_\uparrow + n_\downarrow)\Big]$$

$$= \frac{1}{4}\Big[4S^2 + 4S \Big] = S(S+1).$$

The spin states $|s, m\rangle$ are given as:

$$|s, m\rangle = \frac{(b^{+}_{\uparrow})^{s+m} (b^{+}_{\downarrow})^{s-m}}{\sqrt{(s+m)!}\sqrt{(s-m)!}} |0\rangle$$

One may generalize this rebresentation to the SU(N) group, which will be helpful to develop a mean-field approximation:

Consider SU(N) spin operators

$$S_{\alpha\beta} = b^{+}_{\alpha} b_{\beta}$$

where α, β runs from 1 to N.

These operators satisfy the SU(N) algebra:

$$[S_{\alpha\beta}, S_{\gamma\delta}] = \delta_{\beta\gamma} S_{\alpha\delta} - \delta_{\alpha\delta} S_{\gamma\beta}$$

Check: $[b^{+}_{\alpha} b_{\beta}, b^{+}_{\gamma} b_{\delta}]$

$= b^{+}_{\gamma}[b^{+}_{\alpha}, b_{\delta}] b_{\beta} + b^{+}_{\alpha}[b_{\beta}, b^{+}_{\gamma}] b_{\delta}$

$$= - b^\dagger_\gamma b_\beta \delta_{\alpha\delta} + b^\dagger_\alpha b_\delta \delta_{\beta\gamma}.$$

The above representation corresponds to the fundamental representation of SU(N). For a ferromagnet, we will put all lattice sites in this representation. For the anti-ferromagnet however, one can't use this representation for all sites because of staggered magnetization. It will turn out that one can use fundamental representation for sublattice A and conjugate representation for sublattice B.

Conjugate representation : $\tilde{S}_{\alpha\beta} = b^\dagger_\beta b_\alpha$.

Ferromagnet :

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

first consider the SU(2) case.

$$\vec{S} = b^\dagger \frac{\vec{\sigma}}{2} b$$

$$\Rightarrow H = -\frac{J}{4} \sum_{\langle ij \rangle} \left[b^\dagger_{i\alpha} \sigma^a_{\alpha\beta} b_{i\beta} b^\dagger_{j\gamma} \sigma^a_{\gamma\delta} b_{j\delta} \right]$$

$$= -\frac{J}{4} \sum_{\langle ij \rangle} \left[b^\dagger_{i\alpha} b_{i\beta} b^\dagger_{j\gamma} b_{j\delta} \left(2 \delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\beta} \delta_{\gamma\delta} \right) \right]$$

$$= -\frac{J}{4} 2 \sum_{\langle ij \rangle} b^\dagger_{i\alpha} b_{i\beta} b^\dagger_{j\beta} b_{j\alpha}$$

$$+ \frac{J}{4} \sum_{\langle ij \rangle} b^\dagger_{i\alpha} b_{i\alpha} b^\dagger_{j\gamma} b_{j\gamma}$$

$$= -\frac{J}{2} \sum_{\langle ij \rangle} \left[b^\dagger_{i\alpha} b_{j\alpha} b_{i\beta} b^\dagger_{j\beta} - 2 S(S+1) \right]$$

This expression suggests a large-N approximation where α, β run from 1 to N. Let's denote the mean-field parameters $\langle b^\dagger_{i\alpha} b_{j\alpha} \rangle \equiv Q$. The analog of the constraint $\sum_\alpha b^\dagger_{i\alpha} b_{i\alpha} = 2S$ for SU(N) is $\sum_\alpha b^\dagger_{i\alpha} b_{i\alpha} = NS$.

$$- J\, b^\dagger_{i\alpha} b_{j\alpha} b_{i\beta} b^\dagger_{j\beta}$$

$$\underset{MF}{\simeq} -J \left[Q\, b_{i\beta} b^\dagger_{j\beta} + Q\, b^\dagger_{i\alpha} b_{j\alpha} - Q^2 \right]$$

where we have assumed Q to be real (it doesn't change the result in this problem).

The constraint $\sum_\alpha b^\dagger_{i\alpha} b_{i\alpha} = Ns$ is introduced by a Lagrange multiplier as $\lambda\left(\sum_\alpha b^\dagger_{i\alpha} b_{i\alpha} - Ns\right)$ where we assume λ to be independent of site i. Collecting everything,

$$H_{MF} = -JQ \sum_{\langle ij\rangle,\alpha} \left(b^\dagger_{i\alpha} b_{j\alpha} + h.c.\right)$$

$$+ \lambda \sum_{i\alpha} b^\dagger_{i\alpha} b_{i\alpha}$$

$$+ J Q^2 \underbrace{\frac{VZ}{2}}_{=\#\text{ of bonds}} N - \lambda N V S$$

where V denotes # of sites.

Fourier transforming,

$$H_{MF} = \sum_{k\alpha} \underbrace{(\lambda - JZQ\gamma_k)}_{\varepsilon_k} b^\dagger_{k\alpha} b_{k\alpha}$$

$$+ JQ^2 \frac{VZN}{2} - \lambda SNV$$

The free energy is $F(T = \beta^{-1}) =$

$$N \sum_k \log [1 - e^{-\beta \varepsilon_k}]$$

$$+ VZNQ^2 \frac{J}{2} - \lambda SNV.$$

Minimizing F w.r.t. to λ, Q.

$$\frac{1}{V} \sum_k n_B(\varepsilon_k) = S$$

$$\frac{1}{V} \sum_k n_B(\varepsilon_k) \gamma_k = Q$$

where $n_B(\varepsilon) = \dfrac{1}{e^{\beta\varepsilon} - 1}$ is the Bose fn.

At small $\vec{k}$, $\gamma_{\vec{k}} \simeq 1 - \frac{k^2}{z}$

$$\Rightarrow \quad \varepsilon_k \simeq \lambda - Jz\,a\left(1 - \frac{k^2}{z}\right)$$

$$= \lambda - Jza + Ja\,k^2$$

There are two possibilities (a) $\lambda - Jza > 0$ $\Rightarrow$ the spectrum is gapped. This corresponds to a disordered phase.

(b) $\lambda - Jza = 0$ as $V \to \infty$. This corresponds to ferromagnet with Goldstone modes $\omega_k \simeq k^2$.

As an example, consider $d = 1$. Denoting $\lambda - Jza \equiv JS\kappa^2$

$$\varepsilon(k) \sim JS(k^2 + \kappa^2)$$

At low T, only $\varepsilon \ll T$ contribute, $n_B(\varepsilon) \sim \dfrac{1}{1+\beta\varepsilon - 1} \sim \dfrac{T}{\varepsilon}$

Consider e.g. $d=1$,

$$\frac{1}{2\pi}\int \frac{T\, dk}{JS\,(k^2+K^2)} \simeq S$$

$$\simeq \frac{T}{K\,JS} \simeq S$$

$$\Rightarrow \quad K \sim \frac{T}{JS^2}$$

Using this, one may calculate spin-spin correlation f^{ns} at low-T:

$$\langle s^+(r)\, s^-(r')\rangle$$

$$\simeq \left| \frac{1}{V}\sum_k n_B(k)\, e^{ik\cdot(r-r')} \right|^2$$

$$\simeq \left| \int \frac{T}{JS\,(k^2+\kappa^2)}\, e^{ik\cdot(r-r')} \right|^2$$

$$\sim e^{-|r-r'|/\xi}$$

where $\xi \sim \dfrac{1}{\kappa} \sim \dfrac{JS^2}{T}$.

2+1-d:

$$\frac{T}{JS}\int \frac{d^2k}{k^2+\kappa^2} \simeq S$$

$$\Rightarrow \frac{T}{JS}\log\left(\frac{\Lambda}{\kappa}\right) \simeq S$$

$$\Rightarrow \quad \kappa \sim e^{-JS^2/T}$$

$$\Rightarrow \quad \langle S^+(r)\, S^-(r) \rangle \sim e^{-r/\xi}$$

$$\text{where } \xi \sim \frac{1}{\kappa} \sim e^{+JS^2/T}$$

The exponential dependence on T is a general feature of lower-critical dimension (i.e. dimension at and below which ordering doesn't occur).