Renormalization of Kondo Scale

(Anderson's 'Poor man's RG')

The Kondo Hamiltonian is given by

$$H = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + J_K c^\dagger(\vec{r}=0)\vec{\sigma}\, c(\vec{r}=0)\cdot\vec{S}$$

$$= \sum_{k} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \frac{J_K}{V}\sum_{kk'} c^\dagger(\vec{k})\vec{\sigma}\, c(\vec{k}')\cdot\vec{S}$$

The single impurity acts as an elastic scatterer. We are interested in understanding the renormalization of J_K due to high energy modes. Specifically, consider the term $J_K c^\dagger(k)\vec{\sigma}\, c(k')\cdot\vec{S}$ where k, k' are close to the Fermi surface.

If one integrates out the high energy modes, i.e. modes far away from the Fermi energy, how does J_K renormalizes?

One can answer this dividing e^-s into two groups: $-D < \tilde{\varepsilon}_k < D$ (low energy) and $D < |\tilde{\varepsilon}_k| < D + \delta D$ (high-energy) and considering second order processes where the virtual states belong to the high energy modes and are integrated out.

Here are two terms that contribute.

(a) Contribution from high-energy modes with $D < \hat{\tilde{\varepsilon}}_k < D + \delta D$:

These modes are unoccupied in the ground state. The contribution is

$$J_K^2 \sum_{k'} \langle c^{+}_{k_1} \vec{\sigma} c_{k'} \cdot \vec{S} \quad c^{+}_{k'} \vec{\sigma} c_{k_2} \cdot \vec{S} \rangle$$

where the expectation value $\langle \quad \rangle$ is taken over the high energy modes k' while k_1, k_2 are low-energy modes.

Within the path integral, this contribution is,

$$J_K^2 \left\langle \bar{c}_\alpha(k_1, \omega_1)\, \sigma^a_{\alpha\beta}\, c_\beta(k', \omega_1)\, S^a \right.$$

$$\left. \bar{c}_\gamma(k', \omega_2)\, \sigma^b_{\gamma\delta}\, c_\delta(k_2, \omega_2)\, S^b \right\rangle$$

One may use $\langle c_\beta(k', \omega_1)\, \bar{c}_\gamma(k', \omega_2) \rangle$

$$= \frac{\delta_{\beta\gamma}\, \delta_{\omega_1 \omega_2}}{i\omega_1 - \tilde{\varepsilon}_{k'}} [1 - n_F(\varepsilon_{k'})]$$

$\Rightarrow$ The contribution for on-shell terms i.e. $i\omega_1 = \varepsilon_{k_1} = \varepsilon_{k_2}$ becomes

$$\sum_{k'} c^\dagger_\alpha(k_1)\, c_\delta(k_2)\, \sigma^a_{\alpha\beta}\, \sigma^b_{\beta\delta}\, S^a S^b \frac{[1 - n_F(\varepsilon_{k'})]}{\tilde{\varepsilon}_{k_1} - \tilde{\varepsilon}_{k'}}$$

One may simplify the spin-indices as $\sigma^a_{\alpha\beta}\,\sigma^b_{\beta\gamma}\;S^a S^b$

$$= (\sigma^a \sigma^b)_{\alpha\gamma}\, S^a S^b$$

$$= \left[\delta_{\alpha\gamma}\delta_{ab} + i\,\varepsilon_{abc}\,\sigma^c_{\alpha\gamma}\right]$$

$$\left[\delta_{ab} + i\,\varepsilon_{abc'}\, S^{c'}\right]$$

$$= 2\,\delta_{\alpha\gamma} - 2\,\sigma^c_{\alpha\gamma}\, S^c$$

$\Rightarrow$ The renomalized term is proportional to :

$$-J_K^2\; c^\dagger_{k_1}\vec{\sigma}\, c_{k_2}\cdot\vec{S}\;\sum_{k'}\frac{1}{\left[\tilde{\varepsilon}_{k_1}-\tilde{\varepsilon}_{k'}\right]}$$

Since $\tilde{\varepsilon}_{k_1}\simeq 0$ and $\tilde{\varepsilon}_{k'}\simeq D$, the contribution to δJ_K from this term is $\delta J_K \simeq J_K^2\,\dfrac{N(0)\,\delta D}{D}$

where $N(0)$ is the D.O.S. at the Fermi energy.

(a) Contribution from high-energy modes with $-D > \hat{\tilde{\varepsilon}}_k > -(D+\delta D)$:

$$J_K^2 \Big\langle \bar{c}_\alpha(k', \omega_1)\, \sigma^a_{\alpha\beta}\, c_\beta(k_1, \omega_1)\, S^a$$
$$\bar{c}_\gamma(k_2, \omega_2)\, \sigma^b_{\gamma\delta}\, c_\delta(k', \omega_2)\, S^b \Big\rangle$$

Again, following similar algebra as above one obtains a similar contribution,

$$\delta J_K \simeq N(0) \frac{\delta D}{D} J_K^2 .$$

$$\Rightarrow \quad \frac{dJ_K}{d[\log D]} = 2 J_K^2 \, N(0)$$

or, $\quad \dfrac{dJ_K}{dl} = 2 J_K^2 \, N(0) \quad$ where $\delta l = \dfrac{\delta D}{D}$.

Kondo temperature and length:

The above RG equation seems to indicate that J_K becomes arbitrarily large at low energies. However, one can't trust the leading order RG result when J_K becomes too large compared to other scales. The correct answer is that at low T, one obtains the result advertised earlier that conduction e^-'s scatter unitarily with the impurity spin. The cross-over temperature at which this happens is called 'Kondo temperature). To estimate this, We integrate the above eqn. between $J(l=0) = J_K$ (= bare / microscopic interaction strength) and $J(l \gg 1) \gg J_K$. When $l=0$, the energy scale $\simeq \varepsilon_F$ and when $l \gg 1$, the energy scale $\simeq T_K$, the Kondo temperature. Doing the integration, one obtains,

$$-\frac{1}{J_K(l \gg 1)} + \frac{1}{J_K(l=0)}$$

$$= 2N(0)\, l$$

$$e^{l} = \frac{\varepsilon_F}{T_K} \quad \Rightarrow \quad T_K = \varepsilon_F\, e^{-l}$$

$$= \varepsilon_F\, e^{-\frac{1}{2N(0)J_K}}$$

We note that T_K precisely corresponds to the binding energy $|\Delta_K|$ found earlier using variational approach.

One can similarly define a Kondo screening length:

$$e^{l} = \frac{\xi_K}{a}, \quad a = \text{lattice spacing.}$$

$$\Rightarrow \quad \xi_K = a\, e^{l} = a\, e^{\frac{1}{2N(0)J_K}},$$

again in agreement with the earlier discussion based on the variational wave-fn.