Fermi Liquid Theory

Basic idea: let's start with non-interacting electrons, $H_0 = \sum_k \varepsilon_k c^\dagger_k c_k$, and slowly turn on interactions (we will discuss later what does 'slowly' mean). If this can be done at all, then the ground state will evolve as $|\psi_t\rangle = T\, e^{-\int_{-\infty} i H_I(t)\, dt} |\psi_{-\infty}\rangle$ where H_I is the interaction, T is time-ordering and we have arbitrarily assumed that interactions are turned on at $t = -\infty$ and at $t=0$, we have the desired interacting Hamiltonian,

$$H = H_0 + H_I(t=0)$$

$$\equiv H_0 + H_I.$$

The key idea of Landau Fermi liquid theory is that <u>if</u> the above procedure can be carried out without encountering a phase transition, then the low-lying eigenstates ($\Delta E \sim \frac{1}{L^{\#}}$) of H are in one-to-one correspondence with the eigenstates of H_0. To make this correspondence precise, we note

$$|\psi_{t=-\infty}\rangle = \prod_{k<k_f} c^{\dagger}_{k} |0\rangle$$

$$\Rightarrow \quad |\psi_{t=0}\rangle = \prod_{k<k_f} a^{\dagger}_{k} |0\rangle$$

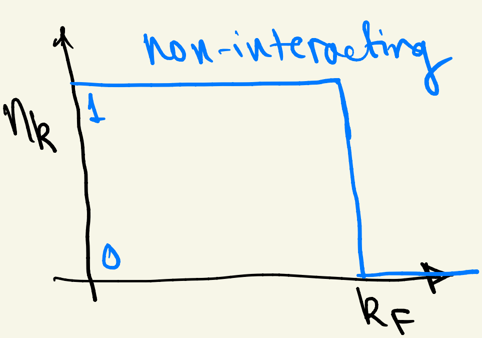
where $a^{\dagger}_{k} = U c^{\dagger}_{k} U^{\dagger}$

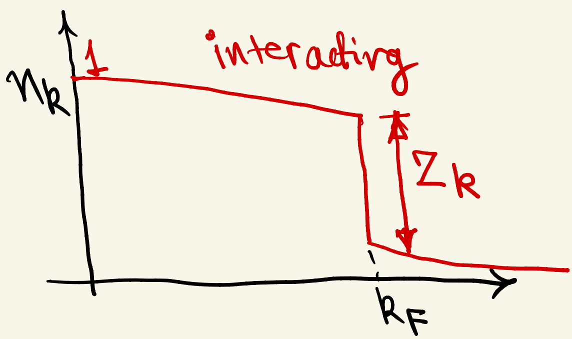
where $U = T e^{-i\int H_I(t') dt'}$.

Fermi liquid theory assumes that $a^\dagger_k$ and $c^\dagger_k$ are related as,

$$c^\dagger_{k\sigma} = \sqrt{Z_k}\, a^\dagger_{k\sigma} + A(\{k\})\, a^\dagger_{k_\sigma} a^\dagger_{k_1\sigma'} a_{k_1\sigma'} + \ldots$$

where crucially, $\sqrt{Z_k} \neq 0$. This implies that $n_k = \langle c^\dagger_{k\sigma} c_{k\sigma} \rangle$ has a discontinuity of Z_k at the Fermi surface.





The definition of a Fermi liquid is that $Z_k \neq 0$.

By construction, the interacting ground state resembles non-interacting ground state of 'renormalized' fermions which are created / destroyed by $a^\dagger$ / a. The low-lying eigenstates correspond to particle-hole excitations created by a / $a^\dagger$. For example, if we denote the interacting ground state $(= \prod a^\dagger_k |0\rangle)$ as $|g.s.\rangle$, then an excited state is given by

$$|\text{excited}\rangle = a^\dagger_{k_1} a_{k_2} |g.s.\rangle$$

where $k_1 > k_F$ and $k_2 < k_F$. The energy of this state, compared to $|g.s.\rangle$ is

$$|\varepsilon_{k_1} - \mu| + |\varepsilon_{k_2} - \mu|$$

where μ is the Fermi energy $= k_F^2/2m$.

These excitations are called <u>quasiparticles</u>.

Clearly, $|excited\rangle$ is in one-to-one correspondence with the non-interacting state $|excited\rangle_0 = c^{\dagger}_{k_1} c_{k_2} |g.s.\rangle$.

The energies ε_{k_1}, however, is not equal to $\varepsilon^0_{k_1}$, the non-interacting energy. This is because interactions will generically renormalize the mass.

$$\varepsilon_{k_1} = \frac{k^2}{2m^*} , \quad \varepsilon^0_{k_1} = \frac{k^2}{2m_e}$$

where m_e is the mass of the e^- and $m^* \neq m_e$ generically.

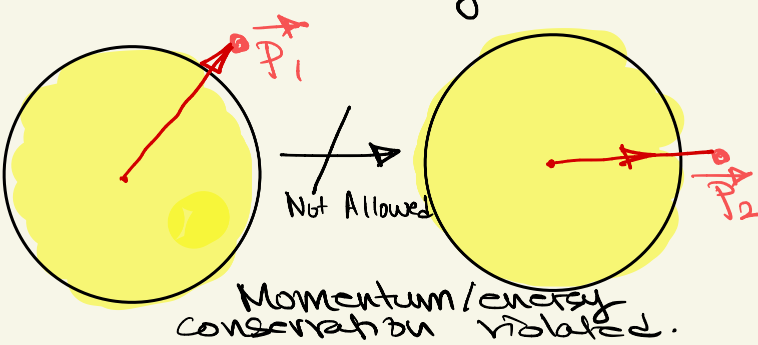
Given above discussion, one might naively expect that <u>all</u> excitations of the interacting system are in one-to-one correspondence with the non-interacting one.

However, one needs to make a distinction between excited eigenstates and excitations that are generated by perturbing eigenstates e.g. sound modes. As we will discuss later, the interacting system has new kinds of sound modes that have no counterpart in the non-interacting system.

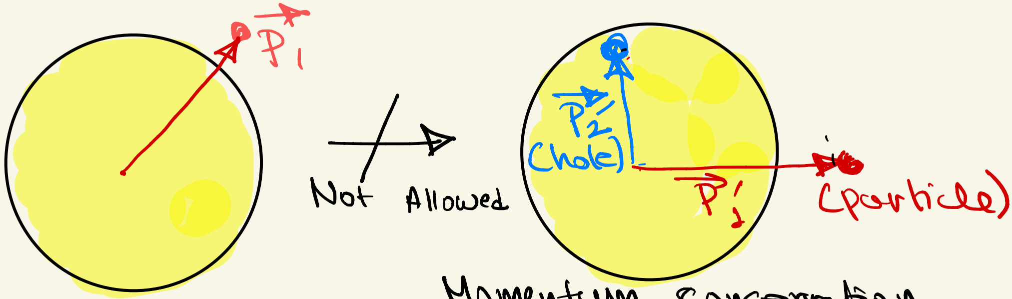
To make progress with questions such as excitations in the interacting system, renormalization of the mass etc., we need to develop an effective theory of interactions between quasiparticles.

To do so, we need to understand allowed scattering processes between the quasiparticles at low energies. Momentum and energy conservation lead to strong constraints on allowed processes.

Consider a single q.p. located outside the Fermi surface, with momentum $\vec{p}_1$. We are interested in the scattering rate for this state to decay into any other allowed state, subject to energy, momentum and charge conservation. Clearly, it can't decay into a single q.p. with momentum $\neq \vec{p}_1$ due to energy momentum conservation.

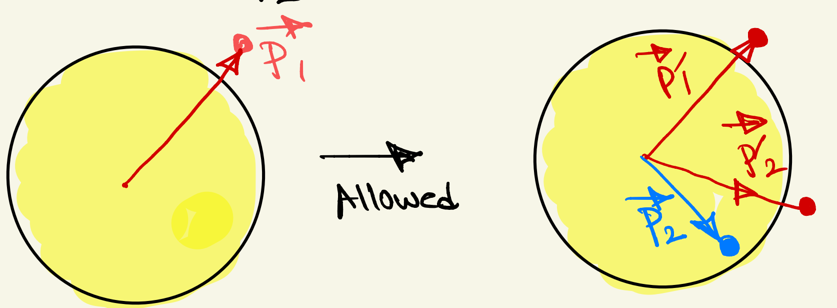


It also can't decay into two particles, or a particle-hole pair either, due to charge conservation.



Momentum conservation satisfied but charge-conservation violated.

The simplest process is when the particle excitation $\vec{p}_1$ decays into three excitations: two particles with momentum $\vec{p}'_1$, $\vec{p}'_2$ and a hole with momentum $\vec{p}_2$.



Momentum-energy conservation requires.

$$\vec{p}_1 = \vec{p}'_1 + \vec{p}'_2 - \vec{p}_2$$

$$p_1^2 = p'^2_1 + p'^2_2 - p_2^2$$

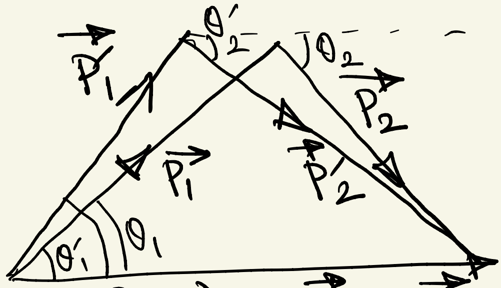
Note the minus sign in front of the contribution from the hole (= absence of particle) contribution.

Furthermore, $\{p_1, p'_1, p'_2\} > p_F$

while $p_2 < p_F$.

Close to $T=0$, all excitations are located close to the Fermi surface, so let's assume that $(p_1 - p_F) \lll p_F$.

Then the other three momenta are also forced to be near p_F due to energy-momentum conservation.



$$\vec{P_1} + \vec{P_2} = \vec{P_1'} + \vec{P_2'}$$

Momentum conservation imply that

$$P_1 \cos(\theta_1) + P_2 \cos(\theta_2) = P_1' \cos(\theta_1') + P_2' \cos(\theta_2')$$

However, since all four vectors are almost equal in magnitude, at the leading order $\theta_1 \simeq \theta_1' = \theta_2 = \theta_2'$

$$\Rightarrow \quad P_1 + P_2 \simeq P_1' + P_2'$$

From Fermi Golden rule, the scattering rate for this process is given by

$$\Gamma \sim \int_{\vec{P_2}, \vec{P_1'}} \delta\left(P_1^2 + P_2^2 - P_1'^2 - P_2'^2\right)$$

where we are integrating only over $\vec{p}_2$ and $\vec{p}'_1$ since $\vec{p}'_2$ is already determined from $\vec{p}_1$, $\vec{p}'_1$ and $\vec{p}_2$. Furthermore, the energy conservation is already accounted for when we above derived $p_1 + p_2 = p'_1 + p'_2$. Apart from various angular coordinate integrals, which just contribute an overall prefactor (see e.g. Coleman's book), Γ is essentially given by $\Gamma \sim \int dp'_1 \, dp_2$ subject to the above derived constraint

$$p_1 + p_2 = p'_1 + p'_2$$

and the requirement $p_1, p'_1, p'_2 > p_F$, while $p_2 < p_F$.

$$p_2' > p_F \Rightarrow \quad p_1 + p_2 - p_1' > p_F$$

$$\Rightarrow \quad p_1' < (p_1 + p_2 - p_F)$$

Further $p_1' > p_F$ as already mentioned.

$$\Rightarrow \quad p_F < p_1' < p_1 + p_2 - p_F .$$

$$\Rightarrow \quad p_2 > 2p_F - p_1$$

Combining this with $p_2 < p_F$, one obtains,

$$2p_F - p_1 < p_2 < p_F .$$

Therefore the integral $\Gamma \sim \int dp_2 \, dp_1'$ is

$$\int_{2p_1 - p_F}^{p_F} dp_2 \int_{p_F}^{p_1 + p_2 - p_F} dp_1' .$$ Clearly this

integral vanishes when $p_1 = p_F$. Doing the integral explicitly, one finds

$$\Gamma \sim \frac{1}{2}(p_1 - p_F)^2 .$$

Therefore, a quasiparticle with energy $e \sim v_F(p_1 - p_F)$ has a lifetime of order $1/e^2$ at $T=0$. This justifies why quasiparticles remain well-defined since e scales as $\frac{1}{L}$ for low-lying excitations, and hence $\Gamma \ll e$. One may also use this result to justify adiabatic preparation of eigenstates starting with non-interacting $\bar{e}$'s. The required to prepare an eigenstate containing a few quasiparticles must be larger than inverse level

Spacing $\sim L$, to avoid complete failure of adiabaticity. At the same time, the preparation time must be less than the quasiparticle lifetime — otherwise quasiparticles disintegrate by the time preparation is complete. Since quasiparticle with energy $e \sim \frac{1}{L}$ has lifetime $O(L^2)$ by above analysis, and $L^2 \gg L$, one has enough time to prepare an eigenstate with quasiparticles adiabatically.

Single-particle Green's function in a Fermi liquid.

The single-particle Green's fn for <u>free fermions</u> is given by

$$G(t,k) = -i \langle g.s. | T c_k(t)\, c^\dagger_k(0) | g.s. \rangle$$

(T = time ordering)

$$= -i \langle g.s. | c_k(t)\, c^\dagger_k(0) | g.s. \rangle \;, \text{ for } t > 0$$

$$= +i \langle g.s. | c^\dagger_k(0)\, c_k(t) | g.s. \rangle, \text{ for } t < 0$$

Note the relative change of sign.

Using $c_k(t) = e^{-it\tilde{\varepsilon}_k} c_k(0)$

and $\langle c^\dagger_k c_k \rangle = n_F(\tilde{\varepsilon}_k) = \theta(-\tilde{\varepsilon}_k)$

$G(t,k)$ for $t > 0$

$$= -i \langle g.s. | c_k\, c^\dagger_k | g.s. \rangle\, e^{-it\tilde{\varepsilon}_k}$$

$$= -i\,[1 - n_F(k)]\, e^{-it\tilde{\varepsilon}_k}$$

$$= -i\, \theta(\tilde{\varepsilon}_k)\, e^{-it\tilde{\varepsilon}_k}$$

Similarly, for $t<0$, $G(t,k)$

$$= +i \langle g.s.|\, c^{\dagger}_k c_k \,|g.s.\rangle\, e^{-it\tilde{\varepsilon}_k}$$

$$= i\, \theta(-\tilde{\varepsilon}_k)\, e^{-it\tilde{\varepsilon}_k}$$

Combining the two expressions into one,

$$G(t,k) = -i\theta(t)\,\theta(\tilde{\varepsilon}_k)\,e^{-it\tilde{\varepsilon}_k} + i\theta(-t)\,\theta(-\tilde{\varepsilon}_k)\,e^{-it\tilde{\varepsilon}_k}$$

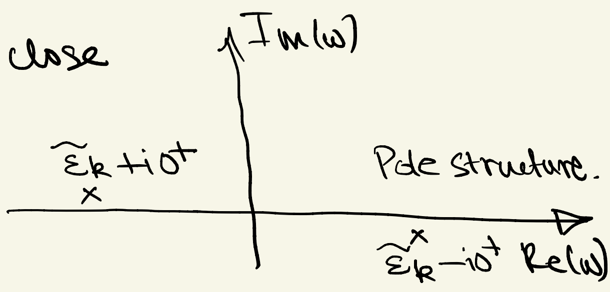
Fourier transforming, $G(\omega,k) = \int dt\, G(t,k)\, e^{i\omega t}$

$$= \frac{1}{\omega - \tilde{\varepsilon}_k + i0^+ \,\mathrm{sign}(\omega)}$$

The $i0^+ \,\mathrm{sign}(\omega)$ takes care of the time ordering. e.g. when $t>0$

$$G(t,k) \sim \int \frac{e^{-i\omega t}\, d\omega}{\omega - \tilde{\varepsilon}_k + i0^+ \,\mathrm{sign}(\omega)}$$

Since $t>0$, we need to close the contour in the lower half plane.



$\Rightarrow$ One will pick up a non-zero contribution only if $\tilde{\varepsilon}_k > 0$. Hence the factor of $\Theta(\tilde{\varepsilon}_k)$ in $G(t, k)$. One can verify the case $t < 0$ similarly.

In the presence of interactions, as we discussed above, the quasiparticles acquire a finite lifetime. This can be accounted for by adding a 'self-energy' term to $G^{-1}(k, \omega)$

$$G^{-1}(k,\omega) = \omega - \tilde{\varepsilon}_k + i\,0^+\,\mathrm{sign}(\omega) \quad \text{(non-interacting)}$$

$$\rightarrow \omega - [\tilde{\varepsilon}_k + \Sigma(k,\omega)] \quad \text{(interacting)}$$

$$= \omega - [\tilde{\varepsilon}_k + \mathrm{Re}(\Sigma(k,\omega)) + i\,\mathrm{Im}(\Sigma(k,\omega))]$$

The location of the fermi surface is determined by the condition

$$\tilde{\varepsilon}_k + \mathrm{Re}(\Sigma(k,\omega=0)) = 0.$$

If the fermi surface remains spherical, then it is not changed due to Luttinger theorem which guarantees that in a Fermi liquid, the volume of the fermi sea is unchanged.

Let's Taylor series expand G^{-1} in $(k-k_F)$ and ω (around $\omega=0$ and $k=k_F$)

$$G^{-1} \approx \omega - \omega \left.\frac{\partial \operatorname{Re}\Sigma(k,\omega)}{\partial \omega}\right|_{\omega=0}$$

$$- (k-k_F)\, \partial_k \left(\hat{\varepsilon}_k + \operatorname{Re}\Sigma(k,0)\right)\Big|_{k=k_F}$$

$$- i \operatorname{Im} \Sigma(k,\omega)$$

$$= Z^{-1}\left[\omega - Z(k-k_F)\,\partial_k\left(\hat{\varepsilon}_k + \operatorname{Re}\Sigma(k,0)\right)\Big|_{k=k_F}\right.$$

$$\left. - i Z \operatorname{Im}\Sigma(k,\omega)\right]$$

where $Z^{-1} = 1 - \frac{\partial}{\partial \omega} \mathrm{Re}\, \Sigma(k_F, \omega)\Big|_{\omega=0}$

Therefore, one may write

$$G^{-1}(k, \omega) = \frac{Z}{\omega - \tilde{\varepsilon}'(k) + \frac{i}{\tau(k, \omega)}}$$

where $\tilde{\varepsilon}'(k) = Z(k-k_F)\frac{\partial}{\partial k}\left[\tilde{\varepsilon}_k + \mathrm{Re}(\Sigma(k,0)\right]_{k=k_F}$

and $\frac{1}{\tau(k,\omega)} = -Z\,\mathrm{Im}(\Sigma(k,\omega))$

Therefore, $\tau(k,\omega)$ denotes the quasiparticle lifetime and $\tilde{\varepsilon}'(k)$ denotes the renormalized energy/band-structure due to interactions.

Z_1 is q.p. weight / residue which captures the overlap of the q.p. with a bare (non-interacting) electron. In general Z is momentum dependent.

Consequences: The above form of G^{-1} determines effective mass, lifetime of q.p. as well as the jump in n_k at the fermi surface.

Effective mass: If effective mass is m^*, then by definition,

$$\frac{k^2 - k_F^2}{2m^*} = \tilde{\varepsilon}'(k,\omega) = Z(k-k_F)\frac{\partial}{\partial k}\left[\tilde{\varepsilon}_k + \mathrm{Re}\,\Sigma(k,0)\right]_{k=k_F}$$

$$\approx \frac{(k-k_F)k_F}{m^*}$$

$$\Rightarrow \frac{k_F}{m^*} = Z\left[\frac{k_F}{m} + \frac{\partial}{\partial k} \mathrm{Re}(\Sigma(k,0))\Big|_{k=k_F}\right]$$

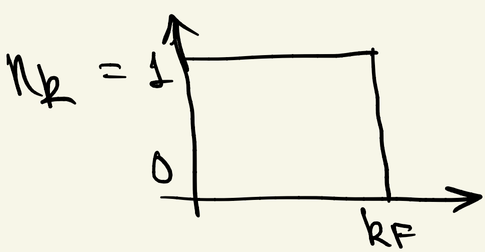
$$\Rightarrow \frac{m}{m^*} = Z\left[1 + \frac{m}{k_F}\frac{\partial}{\partial k}\mathrm{Re}\,\Sigma(k,0)\Big|_{k=k_F}\right]$$

If for some reason, $\frac{\partial}{\partial k}\mathrm{Re}\,\Sigma(k,0) \approx 0$,

then $\frac{m}{m^*} \approx Z$.

Jump in n_k at the Fermi surface:

Recall that for non-interacting e^-s,



i.e. it jumps by one at $k = k_F$.

For the interacting case,

$$G(t,k) = \int \frac{Z_k \, e^{-i\omega t} \, d\omega}{\omega - \tilde{\varepsilon}'(k) + \frac{i}{\tau(k,\omega)}}$$

For simplicity let's assume, $\tau(k,\omega) \equiv \tau(k)$.

For example, as we discussed above, in a fermi liquid, $\tau(k,\omega) \sim \frac{1}{\tilde{\varepsilon}_k^2} \sim \frac{1}{(k-k_F)^2}$

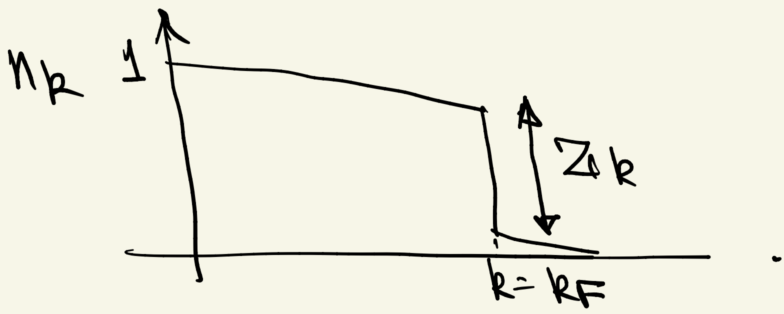
let's consider $G(t, k \approx k_F)$ so that one may neglect $\frac{1}{\tau(k,\omega)}$.

Then $G(t, k \approx k_F) \approx$

$$Z_k \left[-i\theta(t)\,\theta(\tilde{\varepsilon}_k)\,e^{-it\tilde{\varepsilon}_{k_F}} + i\,\theta(-t)\,\theta(-\tilde{\varepsilon}_k)\,e^{-it\tilde{\varepsilon}_{k_F}} \right]$$

when $t = 0^-$, this equals $i\,n(k)$ by definition of $G(t,k)$.

$\Rightarrow$ $n(k)$ jumps by amount Z_k at the Fermi surface.



Quasiparticle lifetime:

$$G(t,k) \sim e^{-t/\tau(k)} \, e^{-it\tilde{\varepsilon}(k)}$$

$\Rightarrow$ quasiparticle lifetime $\simeq \tau(k)$ as also mentioned above.