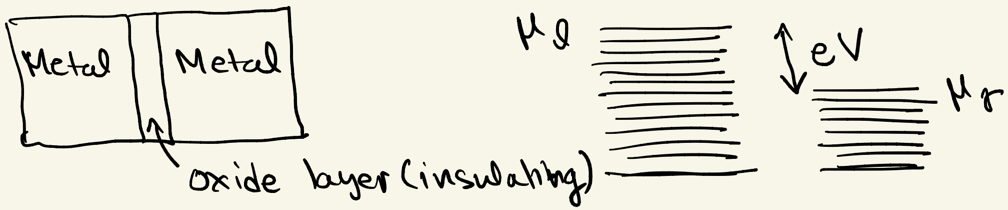
Metal-Metal Junction Vs SC-SC Junction (Josephson Effect)

Metal-Metal Tunnel Junction:



The total Hamiltonian, including the tunneling is

$$H = \sum_k (\varepsilon_{kl} - \mu_l)\, c^\dagger_{kl} c_{kl} + \sum_k (\varepsilon_{kr} - \mu_r)\, c^\dagger_{kr} c_{kr} + \underbrace{\sum_{kk'} \left(c^\dagger_{kl} c_{k'r} T_{kk'} + h.c. \right)}_{= H_T}$$

The current operator may be obtained by its definition and using Heisenberg's EOM:

$$\hat{J} = -e \frac{d\hat{N}_l}{dt} \quad \text{where } \hat{N}_l = \sum_k c^\dagger_{kl} c_{kl}$$

$$= -ei\,[H, \hat{N}_\ell] = -ei\,[H_T, \hat{N}_\ell]$$

$$= ie \sum_{kk'} \left(T_{kk'}\, c^\dagger_{k\ell}\, c_{k'r} - h.c. \right)$$

Note that J is Hermitian.

Let's evaluate $\hat{J}$ perturbatively in T.

When $T=0$, H conserves particle number on the left and right separately $\Rightarrow \langle J \rangle = 0$.

To the first order in T, the ground state is

$$|\varphi\rangle = |\varphi_0\rangle + \sum_{m>0} \frac{\langle \varphi_m | H_T | \varphi_0 \rangle\, |\varphi_m\rangle}{E_0 - E_m + i\delta}$$

due to continuous spectra.

$$\Rightarrow \langle J \rangle \approx \langle \varphi | \hat{J} | \varphi \rangle$$

$$= \sum_{m>0} \left[\frac{\langle \varphi_m | H_T | \varphi_0 \rangle \langle \varphi_0 | \hat{J} | \varphi_m \rangle}{E_0 - E_m + i\delta} \right]$$

$$+ \sum_{m>0} \left[\frac{\langle \varphi_m | \hat{J} | \varphi_0 \rangle \langle \varphi_0 | H_T | \varphi_m \rangle}{E_0 - E_m - i\delta} \right]$$

After plugging in the expression for $\hat{J}, H_T$, the only terms that survive are:

$$\langle J \rangle = e \sum_{kk'm} |T_{kk'}|^2$$

$$\left[|\langle \varphi_0 | c^\dagger_{kl} c_{k'r} | \varphi_m \rangle|^2 - |\langle \varphi_m | c^\dagger_{kl} c_{k'r} | \varphi_0 \rangle|^2 \right]$$

$$\times \left[\frac{1}{E_0 - E_m + i\delta} - \frac{1}{E_0 - E_m - i\delta} \right]$$

The principal part cancels and one obtains the Fermi Golden rule:

$$\langle J \rangle = 2\pi e \sum_{kk'm} |T_{kk'}|^2 \overbrace{\left[|\langle \varphi_0 | c^\dagger_{kl} c_{k'r} | \varphi_m \rangle|^2 - |\langle \varphi_m | c^\dagger_{kl} c_{k'r} | \varphi_0 \rangle|^2 \right]}$$

$$\delta(E_0 - E_m)$$

Lets evaluate the matrix elements under $\overbrace{\quad}$:

$$|\langle \varphi_0 | c^\dagger_{kl} c_{k'r} | \varphi_m \rangle|^2 = n_{kl}(1 - n_{k'r})$$

$$|\langle \varphi_m | c^\dagger_{kl} c_{k'r} | \varphi_0 \rangle|^2 = n_{k'r}(1 - n_{kl})$$

$$\Rightarrow \langle J \rangle = 2\pi e \sum_{kk'} |T_{kk'}|^2 \left[n_F(\tilde{\varepsilon}_k) - n_F(\tilde{\varepsilon}_{k'} + eV) \right]$$

$$\delta(\tilde{\varepsilon}_k - \tilde{\varepsilon}_{k'})$$

Converting sum over k to integral over energy :

$$\langle J \rangle = 2\pi e V_1 V_2 \overline{|T_{kk'}|^2} \int d\tilde{\varepsilon}_k \, d\tilde{\varepsilon}_{k'} \left[n_F(\tilde{\varepsilon}_k) - n_F(\tilde{\varepsilon}_{k'} + eV) \right] N(\tilde{\varepsilon}_k) \, N(\tilde{\varepsilon}_{k'})$$

$$\simeq 2\pi e V_1 V_2 \, eV \, \overline{|T_{kk'}|^2} \, N^2(E_F)$$

where $N(E_F)$ is the density of states at the Fermi level, and we have replaced $|T_{kk'}|^2$ by it's average value.

$\Rightarrow$ the conductance for tunneling between the two metals is

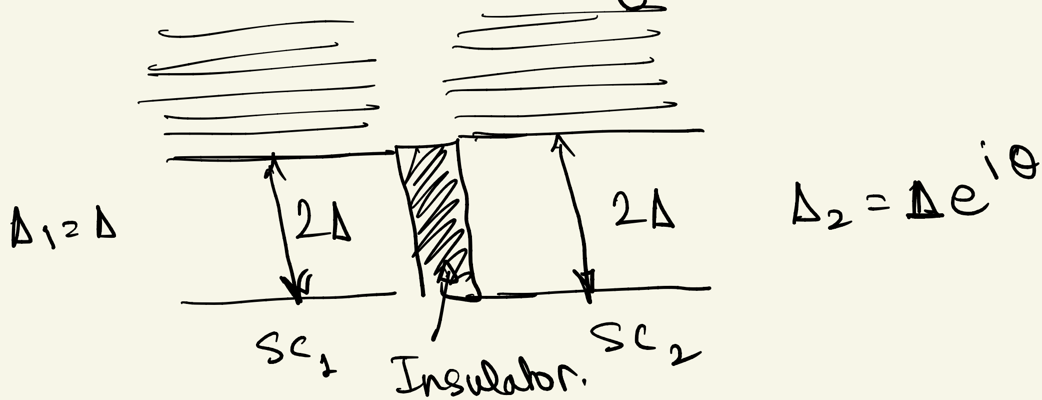
$$G = 2\pi e^2 \, \overline{|T|^2} \, N^2(E_F).$$

For spinful case one would simply gets a factor of two on the RHS.

SC - SC Tunnel Junction

& Josephson effect

Let's first consider SC-SC tunnel junction phenomenologically.



The two superconductors have a phase difference of θ between their order parameters $\Rightarrow$ expect a current even when no voltage applied (recall $J \sim \nabla\theta$)

At energy $\ll \Delta$, the effective Hilbert space consists only of cooper pairs:

$$|m\rangle = |N_L - m, N_R + m\rangle$$

where $|N_L, N_R\rangle$ is some reference state with N_L, N_R number of cooper pairs on the left and right and $|m\rangle$ is the state where m number of cooper pairs have tunneled from left to right.

In the absence of tunneling, the states $|m\rangle$ are all degenerate and essentially eigenstates of the Hamiltonian. Due to tunneling of cooper pairs they will mix

$$H_T = -J \sum_m \left[|m\rangle\langle m+1| + h.c. \right]$$

$$= -J \sum_\varphi \cos(\varphi) |\varphi\rangle$$

where $|\varphi\rangle = \sum_m e^{im\varphi} |m\rangle$

Since $m \sim N_L - N_R$

$\varphi \sim \theta_L - \theta_R$ where θ_L, θ_R are the phase of the order-parameter on left and right

A chemical potential difference couples to $N_L - N_R$ as $2e(N_L - N_R)$.

The total Hamiltonian in the presence of voltage is:

$$H = -J \cos(\hat{\phi}) - 2eV\hat{n}$$

where $\hat{n} = \sum_m m\, |m\rangle\langle m|$ is conjugate to $\hat{\phi}$: $[\hat{\phi}, \hat{n}] = -i$.

Recall our earlier discussion of representing a boson $b = \sqrt{n}\, e^{i\phi}$.

Here $b \sim c^+_\uparrow c^+_\downarrow$.

Let's consider consequences of this Hamiltonian.

$$\frac{\partial \hat{n}}{\partial t} = i[H, \hat{n}]$$

$$= J \sin(\hat{\phi})$$

$$\frac{\partial \hat{\varphi}}{\partial t} = i\,[H, \hat{\varphi}]$$

$$= 2eV$$

$$\Rightarrow \quad \hat{\varphi}(t) = 2eVt + \varphi_0$$

$$\Rightarrow \quad \text{current} = \left\langle \frac{\partial n}{\partial t} \right\rangle \simeq J \sin(\varphi_0 + 2eVt)$$

where φ_0 is the phase difference between the left and right order parameters in the absence of any voltage difference.

DC voltage leads to AC current!

This is very much like Bloch oscillation of e^-'s in the presence of a uniform, static electric field. $H = -t \sum \cos(k) c^+_k c_k$.

$$\frac{dk}{dt} = -eE$$

$$v(t) = \frac{\partial \varepsilon(k)}{\partial k} = t \sin(k)$$

$$= -t \sin(eEt)$$

BCS derivation of Josephson current:

We need to derive an effective Hamiltonian for cooper-pair tunneling.

$$H_{eff} = H_0 + \sum_n \frac{P_n V (1-P_n) V P_n}{H_0 - E_n}$$

$$V = (c^{\dagger}_{k\sigma L} c_{k'\sigma R} T_{kk'} + h.c.)$$

We are interested in the term

$$\sim c^{\dagger}_L c^{\dagger}_L c_R c_R + h.c.$$

$$\frac{\langle BCS | T_{kk'} c^{\dagger}_{Lk\sigma} c_{Rk'\sigma} T_{pp'} c^{\dagger}_{Lp\sigma'} c_{Rp'\sigma'} | BCS \rangle}{\Delta E_{kk'} + \Delta E_{pp'}}$$

ΔE = energy of the state $c^{\dagger}_{Lk} c_{Rk'} |BCS\rangle_L |BCS\rangle_R$

$$\langle BCS_L | c^\dagger_{Lk} \, c^\dagger_{Lp} | BCS \rangle_L \propto \delta_{kp} \, U_k^* V_p$$

$$H_J \sim \sum_{kp} \frac{U_k^* V_k \, U_p V_p^*}{(E_k + E_p)} + h.c.$$

$$\sim N^2(E_F) \int d\varepsilon \, d\varepsilon' \frac{\Delta^2}{(EE')(E+E')} \cos(\varphi) \, \overline{|T|^2}$$

$$E = \sqrt{\varepsilon^2 + \Delta^2}$$

$$\sim N^2(E_F) \, |T|^2 \, \Delta \cos(\varphi)$$

recall, for N-N junction,

conductance $G = 2\pi e^2 \, \overline{|T|^2} \, N^2(E_F)$.

$$\Rightarrow \boxed{E_J \sim G\Delta}$$