Electromagnetic Response of Superconductors

Basics of electron transport:

$$H = \int_x c^+ \frac{(-i\nabla - eA)^2}{2m} c + \dots$$

$$\vec{j} = \frac{-\delta H}{\delta \vec{A}} = \underbrace{- \frac{ie\left[c^+ \vec{\nabla} c - h.c.\right]}{m}}_{\text{paramagnetic current } j_p} \underbrace{- \frac{e^2}{m} \vec{A} c^+ c}_{\text{diamagnetic current } j_d}$$

Only $\vec{j} = \vec{j}_p + \vec{j}_d$ is guage invariant. However, if one chooses a certain guage, e.g. $\nabla \cdot A = 0$, then j_p, j_d separately have physical meaning. In a superconductor in the guage $\nabla \cdot A = 0$ (London guage), j_p vanishes, while j_d is non-zero. This leads to infinite conductivity and Meissner effect. We will discuss this in detail below.

lets consider a specific scenario where one suddenly applies a pulse of electric field at $t=0$: $\vec{A}(t) = A_0 \hat{x}\, \theta(t)$

$$\Rightarrow \vec{E} = -\frac{\partial \vec{A}}{\partial t} = -A_0\, \delta(t)\, \hat{x}.$$ We

assume London guage $(\vec{\nabla} \cdot \vec{A} = 0)$.

$$\langle \vec{j}(t) \rangle = \langle j_p(t) \rangle - \frac{ne^2}{m} A_0\, \theta(t)$$

where we have assumed that electron density is uniform : $c^+ c \to n$.

The diamagnetic part of the current turns on instantaneously and remains non-zero forever. In a normal metal, one expects that the at times $t >$ transport relaxation time, the paramagnetic piece will cancel out the diamagnetic part. (since the electric field is applied only at $t=0$).

In a superconductor however, this cancellation doesn't happen and $\vec{j}_p = 0$ at long times. This leads to non-decaying current forever.

Linear response theory :

Using linear response,

$$\langle j_p^\alpha \rangle(x,t) = i\int_{x',t'} \langle [j^\alpha_p(x,t), j^\beta(x',t')] \rangle A^\beta(x',t') \quad \theta(t-t')$$

The diamagnetic current is already linear in A $\Rightarrow$ $\langle \vec{j}_D(x,t) \rangle = -\frac{ne^2}{m} \vec{A}(x,t)$

Fourier transforming,

$$\langle j^\alpha(\vec{q},\omega) \rangle = \left\{ i \langle [j^\alpha_p(\vec{q},\omega) , j^\beta_p(-\vec{q},-\omega)] \rangle - \frac{ne^2}{m} \delta^{\alpha\beta} \right\} A^\beta(\vec{q},\omega)$$

One may now obtain an expression for the conductivity :

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \Rightarrow \vec{E}(q,\omega) = i\omega \vec{A}(q,\omega)$$

$$\Rightarrow j^{\alpha}(\vec{q},\omega) = \underbrace{\sigma^{\alpha\beta}(\vec{q},\omega)}_{\text{conductivity}} E^{\beta}(\vec{q},\omega)$$

where
$$\boxed{\sigma^{\alpha\beta}_{(\vec{q},\omega)} = \frac{\langle i[\, j^{\alpha}_{p}(q,\omega),\, j^{\beta}_{p}(-q,-\omega)] \rangle}{i\omega} - \frac{ne^2}{m\, i\omega}\delta^{\alpha\beta}}$$

The DC conductivity is obtained ~~at~~

$\lim_{\omega\to 0} \sigma^{\alpha\beta}(\vec{q}=0,\omega)$. Note the order of limits. As claimed above, in a superconductor, in the London guage $(\nabla\cdot A=0)$, $[\, j^{\alpha}_{p}(q,\omega), j^{\beta}_{p}(-q,-\omega)]$ will vanish. Let's show this using BCS theory.

Let's restrict ourselves to the BCS ground state. The paramagnetic part of response is proportional to

$$R_{\alpha\alpha} \stackrel{\text{def}}{=} \int \langle 0 | [\, j^{\alpha}_{p}(q,t) \,,\, j^{\alpha}_{p}(-q,0)] | 0 \rangle \theta(t) e^{i\omega t} dt$$

$$= \int_{t,\omega'} \Big[\sum_n \langle 0 | e^{iHt} j^{\alpha}_{p}(q) e^{-iHt} | n \rangle \langle n | j^{\alpha}_{p}(-q) | 0 \rangle - \sum_n \langle 0 | j^{\alpha}_{p}(-q) | n \rangle \langle n | e^{iHt} j^{\alpha}_{p}(q) | 0 \rangle \Big]$$

$$\frac{e^{i(\omega+\omega')t}}{\omega' - i\varepsilon}$$

$$= \sum_n \left[\frac{|\langle 0 | j^{\alpha}_{p}(q) | n \rangle|^2}{-\omega + E_n - i\varepsilon} - \frac{|\langle n | j^{\alpha}_{p}(q) | 0 \rangle|^2}{-\omega - E_n - i\varepsilon} \right]$$

$$= -\sum_n \left[\frac{|\langle 0 | j^{\alpha}_{p}(q) | n \rangle|^2}{\omega - E_n + i\varepsilon} - \frac{|\langle n | j^{\alpha}_{p}(q) | 0 \rangle|^2}{\omega + E_n + i\varepsilon} \right]$$

To proceed we need the matrix elements $\langle 0 | j^{\alpha}_{p}(q) | n \rangle$ using BCS theory.

$$j^{\alpha}_{p}(q) = \frac{e}{m}\sum_{k\sigma}\left(k_\alpha + \frac{q_\alpha}{2}\right) c^{+}_{k+q,\sigma}\, c_{k,\sigma}$$

$$= \frac{e}{m}\left[\sum_{k}\left(k_\alpha + \frac{q_\alpha}{2}\right)\left[c^{+}_{k+q,\uparrow}\, c_{k,\uparrow} - c^{+}_{-k\downarrow}\, c_{-(k+q)\downarrow}\right]\right]$$

where we have send the dummy variable $k \to -k-q$ for down spins so as to simplify the following calculation.

Using,
$$c^{+}_{k\uparrow} = u_k\, \gamma^{+}_{k\uparrow} + v^{*}_{k}\, \gamma_{-k\downarrow}$$
$$c_{-k\downarrow} = -v_k\, \gamma^{+}_{k\uparrow} + u_k\, \gamma_{-k\downarrow},$$

$$j^{\alpha}_{p}(q) = \frac{e}{m}\sum_{k}\left(k_\alpha + \frac{q_\alpha}{2}\right)\Big\{\overbrace{(u_{k+q}\, u_k + v_{k+q}\, v_k)}^{\mathscr{l}_{k,k+q}}\left(\gamma^{+}_{k+q\uparrow}\gamma_{k\uparrow} - \gamma^{+}_{-k\downarrow}\gamma_{-(k+q)\downarrow}\right)$$
$$+ \underbrace{(u_{k+q}\, v_k - u_k\, v_{k+q})}_{p_{k,k+q}}\left(\gamma^{+}_{k+q\uparrow}\gamma^{+}_{-k\downarrow} + \gamma_{k\uparrow}\gamma_{-k-q\downarrow}\right)\Big\}$$

$\mathscr{l}$ and p are called coherence factors.

At $T=0$, no quasiparticles i.e.

$\gamma_{\uparrow/\downarrow}|0\rangle = 0 \Rightarrow$ the 'd' term does not contribute to the response.

Furthermore, to obtain D.C. response, we have to calculate $\lim_{\omega\to 0} \frac{R_{\alpha\alpha}(\omega, q=0)}{\omega}$

i.e. the limit $q\to 0$ is taken first.

When $q=0$, even the 'p' term vanishes since it involves the combination $(u_{k+q} v_k - u_k v_{k+q})$.

$\Rightarrow$ The paramagnetic contribution to D.C. conductivity vanishes.

$\Rightarrow$ D.C. conductivity $= \lim_{\omega\to 0} \lim_{q\to 0} \sigma_{\alpha\alpha}(q,\omega)$

$$= \lim_{\omega\to 0} \lim_{q\to 0} \frac{\cancelto{0}{R_{\alpha\alpha}}}{\omega} - \frac{ne^2}{m\, i\omega} \delta_{\alpha\beta} = \infty.$$

$\Rightarrow$ Superconductivity!

Meissner effect:

The above calculation also leads to the Meissner effect (i.e. $j \propto A$) although the order of limits is different. We now consider $\lim_{q\to 0} \lim_{\omega\to 0} R_{\alpha\alpha}(q,\omega)$.

Again, the d term doesn't contribute for the same reason as above ($\gamma|0\rangle = 0$). The contribution from the p term is

$\langle n | \gamma^\dagger_{k+q\uparrow} \gamma^\dagger_{-q\downarrow} |0\rangle$ which is non-zero when $|n\rangle = \gamma^\dagger_{k+q\uparrow}\gamma^\dagger_{-q\downarrow}|0\rangle$.

a state whose energy is $E_{k+q} + E_q$.

Plugging this in the expression for $R_{\alpha\alpha}$ one obtains,

$$R_{\alpha\alpha}(q,\omega) = \frac{e^2}{m}\sum_k \left(k_\alpha + \frac{q_\alpha}{2}\right)^2 p^2_{k,k+q}\left[\frac{1}{\omega-(E_k+E_{k+q}+i\varepsilon)} - \frac{1}{\omega+E_k+E_{k+q}+i\varepsilon}\right]$$

Now at $\omega = 0$, as $q \to 0$, $P \to 0$, however the denominator $\to 2E_k$ which is bounded from below by 2Δ, the BCS gap. $\Rightarrow$ $R_{\alpha\alpha}$ vanishes in this limit as well.

$\Rightarrow$ Only the diamagnetic term contributes.

$$\Rightarrow \quad \langle \vec{j} \rangle = -\frac{ne^2}{m} \vec{A}$$

which is the London's eqn. This eqn. is clearly not guage invariant, and is true only in the London guage. Since the current is conserved, it can only respond to the transverse component of the guage field.

In general,

$$j_i(\vec{q}) = -\frac{ne^2}{m}\left[\delta_{ij} - \hat{q}_i \hat{q}_j\right] A_j(\vec{q})$$

which automatically satisfies, $\hat{q}_i j_i = 0$

At finite T, the contribution from p again vanishes. However now the d term gives non-zero contribution.

One can show that,

$$j_i(\vec{q}) = \frac{\rho_s(T) e^2}{m}\left[\delta_{ij} - \hat{q}_i \hat{q}_j\right] A_j(\vec{q})$$

where $\rho_s(T)$ is the superfluid density at temperature T. $\rho_s(T) = \rho_s(0) - \rho_n(T)$ where $\rho_n(T)$ is the 'normal-fluid' density due to quasiparticle excitations. At low $T \ll \Delta$, $\rho_n(T) \sim e^{-\Delta/T}$ while as $T \to T_c$, $\rho_s(T) \to (T_c - T)$.

Distinguishing metal, superconductor and insulator

(Scalapino, White, Zhang PRB 47, 7995 (1993)).

Consider the system on a lattice eg. a tight binding model.

Using linear response,

$$\langle j_x(\vec{q},\omega)\rangle = \left[R_{xx}(\vec{q},\omega) + \langle T_x\rangle \right] A_x(\vec{q},\omega)$$

where $\langle T_x\rangle$ is the kinetic energy along the x-direction $\sim \langle c^\dagger_i c_{i+\hat{x}} + h.c.\rangle$.

To obtain Meissner response, one considers $\omega = 0$, $q_x = 0$ and $q_y \to 0$.

$$\langle j_x(q_x=0, q_y\to 0, \omega=0)\rangle = \frac{\rho_s e^2}{m} A_x(q_x=0, q_y\to 0, \omega=0)$$

where ρ_s is superfluid density.

To obtain D.C. conductivity, one considers $\vec{q}=0$, $\omega\to 0$ limit.

$$\sigma_{DC} = \lim_{\omega\to 0} \frac{\left[i R_{xx}(\vec{q}=0, \omega\to 0) + \langle T_x\rangle \right]}{i\omega}$$

Clean Metal	Disordered Metal or Metal coupled to phonons	Superconductor	Insulator
$\sigma_{DC} = \infty$	$0 < \sigma_{DC} < \infty$	$\sigma_{DC} = \infty$	$\sigma_{DC} = 0$
$\rho_s = 0$	$\rho_s = 0$	$\rho_s \neq 0$	$\rho_s = 0$