Landau–Ginzburg theory at $B \neq 0$.

The SC order parameter $\psi \sim \langle c^{\dagger}_{\uparrow} c^{\dagger}_{\downarrow} \rangle$ carries two units of electrical charge, and therefore, in the presence of an electromagnetic field,

$$F_s = \frac{B^2}{8\pi} + \int d^dx \left[-|r|\,|\psi|^2 + u\,|\psi|^4 + |(i\vec{\nabla} - 2e\vec{A})\psi|^2 \right]$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$.

Deep in the ordered phase, $|\psi|^2 = \frac{|r|}{2u}$.

Let's rescale, $\psi \to \dfrac{\psi}{\sqrt{\frac{|r|}{2u}}}$, so that

$$F_s = \frac{B^2}{8\pi} + \frac{H_c^2}{4\pi} \int d^dx \left[-|\psi|^2 + \frac{|\psi|^4}{2} + \xi^2 |[i\vec{\nabla} - 2e\vec{A}]\psi|^2 \right]$$

where $\dfrac{H_c^2}{4\pi} = \dfrac{|r|^2}{2u}$.

Lets choose $\vec{B} \parallel \hat{z}$ with $\vec{A} = (0, A(x), 0)$.

$$\Rightarrow F_s = \frac{B^2}{8\pi} + \frac{H_c^2}{4\pi} \int d^dx \left[-|\psi|^2 + \frac{|\psi|^4}{2} + \xi^2 \left[|\partial_x \psi|^2 + 4e^2 A^2(x) |\psi(x)|^2 \right] \right]$$

Minimizing F_s w.r.t $\vec{A}$ and ψ^* :

$$-\xi^2 \partial_x^2 \psi + 4\xi^2 e^2 A^2(x) \psi - \psi + |\psi|^2 \psi = 0$$

$$-\frac{1}{4\pi} \frac{d^2 A(x)}{dx^2} + \frac{H_c^2 \xi^2 4 e^2}{4\pi} \times 2 A(x) |\psi(x)|^2 = 0$$

$$\Rightarrow \frac{d^2 A(x)}{dx^2} = 8 e^2 \xi^2 H_c^2 |\psi(x)|^2 A(x)$$

Deep inside the sc, $\psi \sim 1$, $\Rightarrow$

$A(x) \sim e^{-x/\lambda}$ (Meissner effect). where

$\frac{1}{\lambda^2} = 8 e^2 \xi^2 H_c^2$ defines the penetration depth.

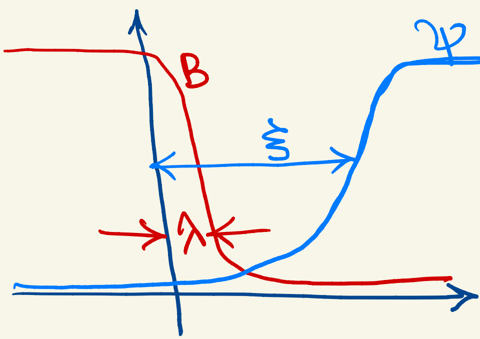
Recall that $\xi \sim \frac{1}{\sqrt{|T_c - T|}}$, and $H_c \sim T_c - T$,

$$\Rightarrow \quad \lambda \sim \frac{1}{\sqrt{T_c - T}}, \Rightarrow \frac{\lambda}{\xi} = O(1).$$

Surface Energy of an N−S interface, and type-I vs type-II SC

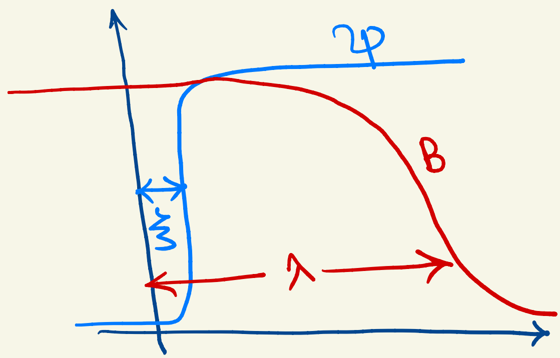
Schematic :

$\lambda \ll \xi$ (Type-I)



Over length scale ξ, flux expulsion but no gain in condensation energy
$\Rightarrow$ Positive surface tension
(i.e. interface energetically unfavorable)

$\lambda \gg \xi$ (Type-II)



Over length scale λ, no flux expulsion, but gain in condensation energy $\Rightarrow$ negative surface tension.

Let's consider applying $H = H_c$ and study the free energy difference between the problem of actual interest and an auxiliary problem where for $x < 0$, $\psi = 0$, $B = H_c$, and for $x > 0$, $\psi = 1$, and $B = 0$.

The free energy density $g = f - \frac{B \cdot H}{4\pi}$ in this auxiliary problem is $-\frac{H_c^2}{8\pi}$ independent of the location: for

$x < 0$, $g = \frac{H_c^2}{8\pi} - \frac{H_c^2}{4\pi} = -\frac{H_c^2}{8\pi}$

for $x > 0$, $g = \frac{H_c^2}{4\pi}\left[\frac{1}{2} - 1\right] = -\frac{H_c^2}{8\pi}$.

The advantage of considering the difference $G - G_{aux}$ is that this contribution comes solely from the interface and therefore can be considered the interface energy per unit cross-section.

$$E_{interface} = G - G_{aux}$$

$$= \frac{B^2}{8\pi} + \frac{H_c^2}{4\pi}\int dx \left[-|\psi|^2 + \frac{|\psi|^4}{2} + \xi^2 \left[|\partial_x \psi|^2 + 4e^2 A^2(x) |\psi(x)|^2 \right] \right.$$

$$- \frac{BH_c}{4\pi} + \frac{H_c^2}{8\pi}$$

$$= \frac{H_c^2}{4\pi}\int dx \left[-\psi^2 + \frac{\psi^4}{2} + \xi^2 [(\partial_x \psi)^2 + 4e^2 A^2(x)\psi^2(x)] \right] + \frac{(B - H_c)^2}{8\pi}$$

where we have further assumed that ψ is real.

Since ψ satisfies Landau–Ginzburg eqn:

$$-\xi^2 \partial_x^2 \psi + 4\xi^2 e^2 A^2(x)\psi - \psi + \psi^3 = 0$$

One may write

$$E_{\text{interface}} = E_{\text{interface}} - \frac{H_c^2}{4\pi}\int dx\, \psi\, [\text{LHS of above Eqn.}]$$

which cancels out the derivative term

$\Rightarrow E_{\text{interface}}$

$$= \frac{H_c^2}{8\pi}\int dx \left[\left(1 - \frac{B}{H_c}\right)^2 - \psi^4\right].$$

Lets consider the two limits,

$\frac{\lambda}{\xi} \ll 1$ (type-I) and $\frac{\lambda}{\xi} \gg 1$ (type-II).

$\frac{\lambda}{\xi} \ll 1$:

In this limit, A rapidly goes to zero, hence $-\xi^2 \partial_x^2 \psi - \psi + \psi^3 \approx 0$,

whose soln. is $\psi = \tanh\left(\frac{x}{\xi\sqrt{2}}\right)$.

$$\Rightarrow E_{\text{interface}} \approx \frac{H_c^2}{8\pi} \int \left[1 - \psi^4\right] dx$$

$$\approx \frac{H_c^2}{8\pi} \xi .$$

Therefore, the N-S interface costs energy in this case.

$\frac{\lambda}{\xi} \gg 1$:

In this case, $A(x) \sim e^{-x/\lambda}$,

and $\psi \sim 1 - e^{-x/\xi}$.

$$\Rightarrow \quad E_{interface} \approx \frac{H_c^2}{8\pi} \times -1 \times \lambda$$

$$\approx -\frac{\lambda H_c^2}{8\pi}$$

Therefore, as a fn of $\frac{\lambda}{\xi}$, $E_{interface}$ changes sign. One can show that $E_{interface} = 0$ at $\frac{\lambda}{\xi} = \frac{1}{\sqrt{2}}$, which therefore serves as the boundary between the type-I vs type-II behavior.