Landau-Ginzburg Theory: Introduction

Let's first review Landau-Ginzburg theory for the Ising model. Denoting the coarse-grained order parameter as ψ,

Landau free energy

$$f = \int d^dx \left[r\, \psi^2(x) + K (\nabla \psi)^2 + u\, \psi^4(x) \right]$$

Assuming $r = a(T - T_c)$ with $a > 0$, deep in the ordered phase $(T \ll T_c)$, one may neglect fluctuations of ψ, and $\psi(x) \approx \psi_0 = \sqrt{\frac{-r}{2u}}$.

Similarly, deep in the paramagnet phase, $\psi_0 = 0$.

Close to the transition, on the paramagnetic side, within Gaussian approximation,

$$f \approx \int d^dx \left[r \psi^2 + (\nabla \psi)^2 \right]$$

$$= \int d^dk \, |\psi(k)|^2 \left[r + k^2 \right]$$

Similarly, approaching from the ordered side, $f \approx \int d^dx \left[r [\psi_0 + \delta\psi]^2 \right.$

$$\left. + u [\psi_0 + \delta\psi]^4 + (\nabla \psi)^2 \right]$$

$$\approx \int d^dx \left\{ [2r + 12u\psi_0^2](\delta\psi)^2 + \nabla(\delta\psi)^2 \right\}$$

$$= \int d^dk \left[-4r + k^2 \right] |\delta\psi(k)|^2.$$

Therefore, in either case,

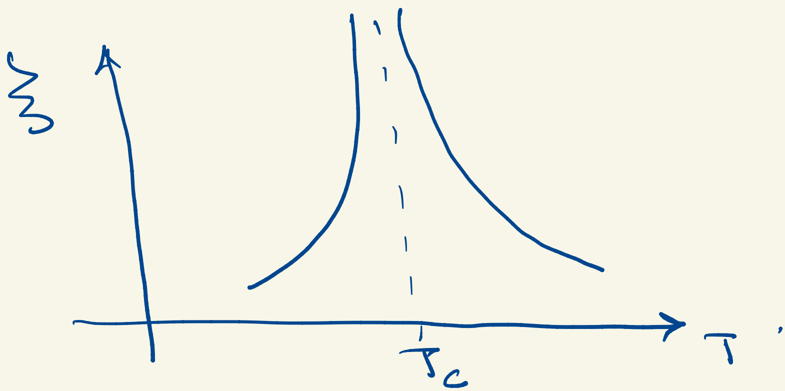
$$f \approx \int d^dx \, |\delta\psi(k)|^2 \left[b|r| + k^2 \right]$$

where $b > 0$.

$$\Rightarrow \langle \delta\psi(r)\,\delta\psi(0)\rangle$$

$$\sim \int d^d k \, \frac{e^{i\vec{k}\cdot\vec{r}}}{b|r| + k^2} \sim \frac{e^{-r/\xi}}{r^{d-2}}$$

where $\xi \sim \dfrac{1}{\sqrt{|r|}} \sim \dfrac{1}{\sqrt{(T-T_c)}}$



Superfluid vs Superconductor

As discussed earlier in the context of interacting neutral bosons, a superfluid breaks the global symmetry corresponding to particle number conservation, $b \to b e^{i\theta}$. Correspondingly, there exists a single Goldstone mode in a superfluid, with dispersion $\varepsilon(k) \sim |\vec{k}|$. In a superconductor, the order parameter $\langle c^{+}_{\uparrow} c^{+}_{\downarrow} \rangle$ is again a complex number, similar to superfluid, and one might naively expect that it will again support a Goldstone mode at low energies. This expectation however is incorret — there is no Goldstone mode in a SC, because there is a crucial difference between a SC

And a SF: the SC order parameter $\langle c^\dagger_\uparrow c^\dagger_\downarrow \rangle$ carries charge-2 of the electromagnetic gauge field, while the SF order parameter $\langle b \rangle$ we studied was charge neutral. In fact, as we will now discuss, a SC <u>does not</u> break any global symmetry spontaneously. This is because there are no global symmetries in a SC — the particle no. conservation symmetry is 'gauged', which is a fancy way of saying that it is not even there to begin with. Let's do a concrete calculation, using Landau-Ginzburg theory, to understand these points.

Landau-Ginzburg theory for a superfluid.

Let's denote the complex order parameter as φ. The Landau-Ginzburg free energy functional is,

$$F[\varphi] = \int d^dx \left[|\nabla\varphi|^2 + \overbrace{(|\varphi|^2 - \varphi_0^2)^2}^{V(\varphi)} \right]$$

In the symmetric (i.e. non-superfluid) phase, $\varphi_0 = 0$ so that the minima of the potential $V(\varphi)$ lies at $\varphi_0 = 0$.

In the superfluid phase, $\varphi_0 \neq 0 \Rightarrow$

$|\varphi| = \varphi_0 e^{i\theta(x)}$ minimizes $V(\varphi)$. To minimize f, in the ground state, $\theta(x)$ is independent of $\vec{x}$, so that $\nabla\varphi = 0$ and $f = 0$.

Crucially, configurations corresponding to different values of θ are physically distinct and correspond to distinct ground states. Of course, $\theta \equiv \theta + 2\pi$, therefore different ground states can be labeled by $e^{i\theta}$.

Fluctuations around the ground states:

To study fluctuations around the ground states, we let $\theta(x)$ to be function of $\vec{x}$: $\varphi(x) = \varphi_0 e^{i\,\theta(x)}$,

so that $F \sim \int d^d x \, (\partial_\mu \theta)^2$.

This is precisely the action for a linearly dispersing Goldstone mode, and therefore, the many-body spectrum is gapless with gap $\sim \frac{1}{L}$ (L = system size).

Landau-Ginzburg theory for a superconductor

The order-parameter is now charged complex scalar and the Landau-Ginzburg free energy is

$$F = \int d^dx \left[|(\partial_\mu - iqA_\mu)\phi|^2 + (|\phi|^2 - \phi_0^2)^2 + (\vec{\nabla}\times\vec{A})^2 \right]$$

where 'q' is the charge carried by the order-parameter ϕ ($q=2$ for a BCS SC).

In the ground state, all three terms in F should evaluate to zero $\Rightarrow$

$$|\phi| = \phi_0 \,, \quad \vec{\nabla}\times\vec{A} = 0 \,, \quad (\partial_\mu - iqA_\mu)\phi = 0$$

$\Rightarrow$ $A_\mu = \partial_\mu \alpha$ ('Pure guage'),

$$\phi = \phi_0 e^{i\alpha q}.$$

The crucial point to note is that different values of the parameter do not correspond to physically distinct configurations. This is because different values of α are related to each other via gauge transformation.

For example, as $\phi \to \phi e^{iq\theta}$,

$A_\mu \to A_\mu + \partial_\mu \theta$. Gauge transformations do not change physical configuration and there gauge 'symmetry' is not a symmetry but simply a redundant description of the same physical state using different set of variables. Therefore, unlike SF, where we had a continuum of ground states labeled by the angle

$e^{i\theta}$, in a SC, there is a unique ground state that does not break any symmetry whatsoever. Correspondingly we do not expect a Goldstone mode in a SC, as we will now verify by an explicit calculation.

To study the fluctuations around the ground state, let us write

$$\phi = \phi_0 + \psi(x) + i\theta(x)$$ where

$\psi(x)$ and $\theta(x)$ are assumed to be small. We now expand F in ψ and θ upto quadratic order (note that a term such as $\psi^2\theta^2$ is quartic order in fluctuations and therefore, will not be considered).

The potential energy term becomes.

$$(|\phi|^2 - \phi_0^2)^2$$

$$= [(\phi_0 + \psi)^2 + \theta^2 - \phi_0^2]^2$$

$\simeq \quad \phi_0^2\, \psi^2$ after dropping higher order terms.

Similarly, $|(\partial_\mu - iqA_\mu)\phi|^2$

$$= |\partial_\mu \psi + i\,\partial_\mu \theta - iqA_\mu(\phi_0 + \psi + i\theta)|^2$$

$$\simeq (\partial_\mu \psi)^2 + (\partial_\mu \theta - qA_\mu \phi_0)^2$$

$$= (\partial_\mu \psi)^2 + q^2 \phi_0^2 [\underbrace{A_\mu - \frac{\partial_\mu \theta}{q\phi_0}}_{B_\mu}]^2$$

$$\equiv (\partial_\mu \psi)^2 + q^2 \phi_0^2\, B_\mu B_\mu .$$

Since B_μ and A_μ are related by a gauge transformation,

$$(\vec{\nabla} \times \vec{A})^2 = (\vec{\nabla} \times \vec{B})^2 .$$

$\Rightarrow$ $f = (\partial_\mu \psi)^2 + \phi_0^2 \psi^2 + q_r^2 \phi_0^2 B_\mu^2$

$+ (\vec{\nabla} \times \vec{B})^2$ upto quadratic order in fluctuations around the g.s.

$\Rightarrow$ ψ has a mass ϕ_0^2, and $\vec{B}$ has a mass $q_r^2 \phi_0^2$.

$\Rightarrow$ No Goldstone mode!

The spectrum is fully gapped with a gap $\sim \phi_0^2$.