This is called a Kramers-Kronig relation.

By definition, $\chi(\omega) = \chi'(\omega) + i\chi''(\omega)$ where $\chi'(\omega)$ and $\chi''(\omega)$ are the real and imaginary parts of $\chi(\omega)$.

Since $\dfrac{1}{x - i\varepsilon} = P\left(\dfrac{1}{x}\right) + i\pi\,\delta(x)$

where P denotes the principal part,

$$\Rightarrow \quad \chi(\omega) = \frac{1}{\pi} P \int \frac{\chi''(\omega')\,d\omega'}{\omega' - \omega} + i\chi''(\omega')$$

Therefore, $\chi'(\omega) = \dfrac{1}{\pi} P \displaystyle\int \frac{\chi''(\omega')\,d\omega'}{\omega' - \omega}$

which is another way to write the Kramers-Kronig relation. This relation is experimentally quite useful if one has access only to (say) the imaginary part of a response function as a function of frequency. Then, one may be able to obtain the real part using above relation.

Linear Response to density perturbations for non-interacting electrons.

Consider the following setup :

$$H(t) = \sum \varepsilon_k c^\dagger_k c_k + \sum_r V(r,t)\, c^\dagger(r) c(r)$$

We turn on $V(r,t)$ smoothly at $t = t_0 (= -\infty)$ and we are interested in change in local density ($= \delta\rho$) at arbitrary times due to this change in the Hamiltonian.

To linear order in V, we know the answer from our discussion above,

$$\delta\rho(r,t) = \int_{t',r'} \chi(r-r', t-t')\, V(r',t')$$

where $\chi(r,t) = -i\,\theta(t)\langle[\rho(r,t), \rho(0,0)]\rangle$

Fourier transforming,

$$\delta\rho(k,\omega) = \chi(k,\omega)\, V(k,\omega)$$

where $\chi(k,\omega)$

$$= -i \int \theta(t) \langle [\rho_k(t), \rho_{-k}(0)] \rangle e^{i\omega t} dt$$

Assuming that initially (i.e. before V is turned on), the system is in the ground state of $H_0 = \sum \varepsilon_k c^+_k c_k$, one can now explicitly calculate the above expectation value using Wick's theorem.

Using $\rho_k = \int c^+(r) c(r) e^{ik.r}$

$$= \frac{1}{V} \sum_q c^+_q c_{q+k}$$

After a bit of algebra, one finds,

$$\chi(k,\omega) = \frac{-i}{V} \int_t \theta(t) e^{i\omega t + i(\varepsilon_q - \varepsilon_{k+q})t} \sum_q [n(\varepsilon_q) - n(\varepsilon_{k+q})]$$

Using $\theta(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\omega' t}}{\omega' - i\varepsilon} d\omega'$, this becomes

$$\chi(k,\omega) = \frac{1}{V} \sum_q \frac{n(\varepsilon_q) - n(\varepsilon_{k+q})}{\omega + i\eta + \varepsilon_q - \varepsilon_{q+k}} \quad (\eta = 0^+)$$