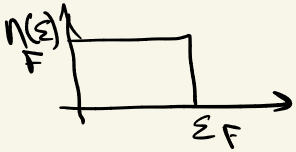
Consequences of Sharp Fermi Surface for correlation f^ns.

At $T=0$, the Fermi-Dirac distribution for non-interacting fermions is a step f^n :



. This singularity in n_F results in power-law correlations, both in space and time.

Equal space, Unequal time correlations :

First consider $\langle c^\dagger(x,t)\, c(x,0)\rangle$ in the ground state.

$$\langle c^\dagger(x,t)\, c(x,0)\rangle = \frac{1}{V}\sum_{kk'}\langle c^\dagger(k,t)\, c(k',t=0)\rangle\, e^{-ik\cdot x + ik'\cdot x}$$

From equation of motion for free electrons,

$$c(k,t) = e^{-i\varepsilon_k t}\, c(k,t=0)$$

$$\Rightarrow \langle c^\dagger(x,t)\, c(x,0)\rangle = \frac{1}{V}\sum_{kk'}\langle c^\dagger(k)\, c(k')\rangle e^{i\varepsilon_k t}\, e^{ix\cdot(k'-k)}$$

where $c^\dagger(k) \equiv c^\dagger(k, t=0)$ as usual.

Further, since $H = \sum_k \varepsilon_k c^\dagger_k c_k$ has the symmetry $c_k \to e^{i\theta_k} c_k$ where θ_k can depend on k, $\langle c^\dagger_k c_{k'} \rangle = 0$ when $k \neq k'$.

$$\Rightarrow \langle c^\dagger(x,t)\, c(x,0) \rangle = \frac{1}{V} \sum_k \langle c^\dagger_k c_k \rangle e^{i\varepsilon_k t}$$

$$= \frac{1}{V} \sum_k n_F(k) e^{i\varepsilon_k t} = \frac{1}{(2\pi)^d} \int d^d k \; n_F(k) e^{i\varepsilon_k t}$$

$$= \frac{1}{(2\pi)^d} \int_0^{\varepsilon_F} d\varepsilon \; N(\varepsilon)\, e^{i\varepsilon t}.$$ The integral will be dominated by $\varepsilon \approx \varepsilon_F$ due to sharp cutoff at $\varepsilon = \varepsilon_F$. Writing $\varepsilon = \varepsilon_F + \delta\varepsilon$,

$$= \frac{1}{(2\pi)^d} \int d(\delta\varepsilon)\; N(\varepsilon_F + \delta\varepsilon)\, e^{i\varepsilon_F t}\, e^{i\delta\varepsilon t}$$

For large t, due to oscillating term $e^{i\delta\varepsilon t}$, the dominant contribution will come only from $|\delta\varepsilon\, t| \lesssim O(1)$ i.e. $-\frac{1}{t} \lesssim \delta\varepsilon \lesssim \frac{1}{t}$

Thus we find that at large t,

$$\langle c^\dagger(x,t)\, c(x,t=0)\rangle \sim \frac{N(\varepsilon_F)}{t}\, e^{it\varepsilon_F}$$

Unequal space, equal time correlations :

Next consider, $\langle c^\dagger(x)\, c(x')\rangle$. Following similar steps as above, this is proportional to

$$\frac{1}{V}\sum_k n_F(\vec{k})\, e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}$$

Due to $n_F(\vec{k})$ being a step fn. the sum over $\vec{k}$ is restricted to $|\vec{k}|<k_F$. Lets work in d-dimensions, with a spherical Fermi surface. The above sum is

$$\frac{1}{(2\pi)^d}\int d^dk\; e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}$$

Lets choose $(x'-x)$ to lie along the

direction "$\hat{x}_1$" in d-dimensions, where the spherical coordinates are defined as,

$x_1 = r\cos(\varphi_1)$, $x_2 = r\sin(\varphi_1)\cos(\varphi_2)$,

$x_3 = r\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3)$,

..... $x_d = r\sin(\varphi_1)\ldots\sin(\varphi_{d-1})$.

The volume element in d-dimensions is

$$d^d V = r^{d-1}\sin^{d-2}(\varphi_1)\sin^{d-3}(\varphi_2)\ldots\sin(\varphi_{d-2})\, dr\, d\varphi_1\, d\varphi_2 \ldots d\varphi_{d-1}$$

Due to this choice, the integral becomes

$$\frac{C}{(2\pi)^d}\int_0^{k_F} dk\, k^{d-1}\, d\varphi_1\, e^{ik|\vec{x}-\vec{x}'|\cos(\varphi_1)}\,[\sin(\varphi_1)]^{d-2}$$

where C is a constant that results from integration over the angular variables $\varphi_2, \varphi_3, \ldots \varphi_{d-1}$.

Again, when $|\vec{x}-\vec{x}'|$ is large compared to k_F^{-1} ($\sim$ inter-electron distance), the integral will be dominated by $k \approx k_F$.

Due to oscillating factor $e^{ik|x-x'|\cos(\varphi_1)}$ the integral over φ_1 will be dominated by $\frac{d\cos(\varphi_1)}{d\varphi_1} = 0$ i.e. $\varphi_1 = 0, \pi$.

Let's first consider contribution from $\varphi_1 \approx 0$. One may approximate $\sin(\varphi_1) \sim \varphi_1$

$$\approx \int dk\, k^{d-1}\, d\varphi_1\, e^{ik|\vec{x}-\vec{x}'|[1-\frac{\varphi_1^2}{2}]} \varphi_1^{d-2}$$

Doing the Gaussian integral over φ_1

$$\approx \int_0^{k_F} dk\, k^{d-1} \frac{e^{ik|\vec{x}-\vec{x}'|}}{[ik|\vec{x}-\vec{x}'|]^{(d-1)/2}}$$

Since the dominant contribution will come from $k \approx k_F$, we can set $k \simeq k_F$ everywhere except the oscillating term $e^{ik|\vec{x}-\vec{x}'|}$. Writing $k = k_F + \delta k$, due to oscillating term, only $|\delta k| \lesssim \frac{1}{|x-x'|}$

will give important contribution.

$\Rightarrow$ The integral becomes,

$$\frac{k_F^{d-1} \; e^{i k_F |\vec{x}-\vec{x}'|}}{i\,|\vec{x}-\vec{x}'|\,\left(i\, k_F |\vec{x}-\vec{x}'|\right)^{(d-1)/2}} .$$

The contribution from $\varphi_1 \simeq \pi$ will just be the complex conjugate of the above.

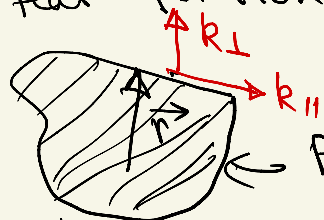
$\Rightarrow \langle c^\dagger(x)\, c(x')\rangle$

$$\approx \frac{\cos\left(k_F |x-x'| - \frac{\pi}{4}(d+1)\right)}{|x-x'|^{(d+1)/2}} .$$

Special case of Nested Fermi surface

As discussed above, for a generic (non-nested) fermi surface, $\langle c^\dagger(r)\, c(0)\rangle \sim \dfrac{\cos(k_F r + \varphi)}{r^{(d+1)/2}}$.

However, for the case of a fermi surface with a flat portion (e.g. nested fermi surface for square lattice at half-filling), $\langle c^\dagger(\vec{r})\, c(0)\rangle$ decays slower if $\vec{r}$ is perpendicular to the flat portion of the fermi surface:



To see this, notice that

$$\langle c^\dagger(\vec{r})\, c(\vec{0})\rangle \sim \int \langle c^\dagger_k c_k\rangle e^{-ik\cdot\vec{r}}\, d^d k$$

$$\sim \int dk_\perp\, d^{d-1}k_\parallel\, e^{-ik_\perp r}$$

$$\sim \frac{e^{-ik_F r}}{r}$$ since the integral over $k_\parallel$ can be done separately and system essentially behaves like a 1d system