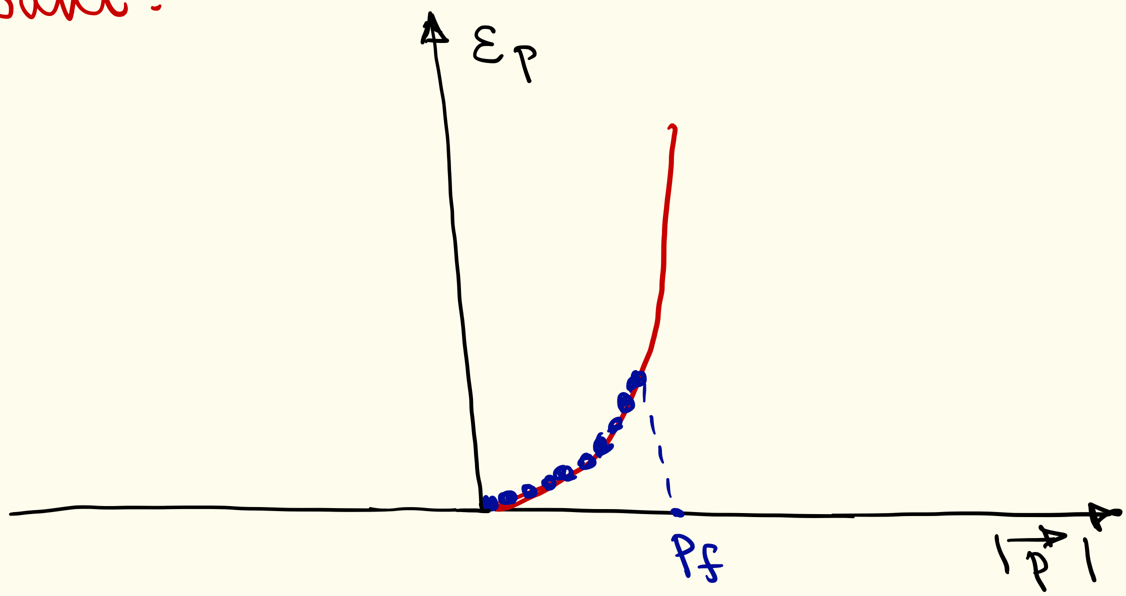
Ground state without taking spin into account:



If there are N fermions (we ignore spin for the moment), they all occupy distinct momenta such that the energy is minimized. Thus, these momenta are contiguous in the energy space.

$$\boxed{\underbrace{|\psi_0\rangle}_{\text{ground state}} = \prod_{|\vec{p}|<p_f} a^{+}(\vec{p}) \underbrace{|0\rangle}_{\substack{\text{No particles} \\ \text{"vacuum"}}}}$$

The crucial thing to note is that p_f is a f^n of particle density and <u>not</u> just particle number.
This is obvious on dimension grounds:

$$p_f \sim \frac{\hbar}{\text{inter-particle distance}} \sim \frac{\hbar}{[V/N]^{1/d}}$$

$$\sim \hbar \, \rho^{1/d}$$

Lets do a more precise calculation:

Number of particles N

$$= \langle \varphi_0 | \hat{N} | \varphi_0 \rangle$$

$$= \langle \varphi_0 | \sum_{\vec{p}} a^+_{\vec{p}} \, a_{\vec{p}} | \varphi_0 \rangle$$

Now $\langle \varphi_0 | a^+_{\vec{p}} \, a_{\vec{p}} | \varphi_0 \rangle = 0$ if $|\vec{p}| > |\vec{p}_f|$

$= 1$ if $|\vec{p}| \leq |\vec{p}_f|$

$$\Rightarrow \quad N = \sum_{|\vec{p}| < p_f} 1$$

$$= V \int_0^{p_f} \frac{d^d p}{(2\pi)^d}$$

$$= V \frac{p_f^d}{(2\pi)^d} \times \text{Volume of } d\text{-dimensional hypersphere of unit radius.}$$

$$= \frac{V p_f^d}{(2\pi)^d} \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}$$

$$\Rightarrow \quad p_f \propto \left(\frac{N}{V}\right)^{1/d} \quad \text{as expected}$$

When $d = 3$

$$N = \frac{V}{8\pi^3} p_f^3 \frac{4}{3}\pi$$

$$\Rightarrow \quad p_f = \left[6\pi^2 \rho\right]^{1/3}$$

where $\rho = N/V$

With Spin:

Let's study ground state of e^- s which have spin-1/2.

Now, the creation/annihilation operators carry two quantum numbers, $\vec{p}$ and $\sigma = \pm 1/2$ i.e. $a^+(\vec{p}, \sigma)$

For free electrons, the energy eigenvalues are independent of spin:

$$H = \sum_{\vec{p}\,\sigma} \frac{p^2}{2m} a^+(\vec{p}, \sigma)\, a(\vec{p}, \sigma)$$

Thus, the ground state is:

$$\boxed{\underbrace{|\psi_0\rangle}_{\text{ground state}} = \prod_{\substack{|\vec{p}| < p_f \\ \sigma}} a^+(\vec{p}, \sigma) \underbrace{|0\rangle}_{\text{No Particles}}}$$

$$N = \sum_{\vec{p},\sigma} \langle \psi_0 | a^\dagger(\vec{p},\sigma)\, a(\vec{p},\sigma) | \psi_0 \rangle$$

$$= \sum_{\sigma=\pm 1/2} \sum_{\vec{p}} \langle \psi_0 | a^\dagger(\vec{p},\sigma) a(\vec{p},\sigma) | \psi_0 \rangle$$

$$= 2 \times \sum_{|\vec{p}|<p_f} 1$$

$$= \frac{V\, p_f^3}{3\pi^2} \quad \text{in} \quad d = 3.$$

$$\Rightarrow \quad p_f = \left[3\pi^2 \rho \right]^{1/3}$$

Ground State Energy:

$$E_0 = \langle \psi_0 | \hat{H} | \psi_0 \rangle$$

$$= \langle \psi_0 | \sum_{\vec{p}\sigma} a^\dagger(p,\sigma) a(p,\sigma) \frac{p^2}{2m} | \psi_0 \rangle$$

$$= \sum_{\sigma} \sum_{|\vec{p}|<p_f} \frac{p^2}{2m}$$

$$= 2 \frac{V}{(2\pi)^3} \int_0^{p_f} \frac{p^2}{2m} 4\pi p^2 \, dp$$

$$= \left[\frac{p_f^2}{2m} \right] \frac{1}{\pi^2} \frac{p_f^3}{5} V$$

Using $\frac{p_f^3 V}{3\pi^2} = N$ from above, one obtains,

$$\frac{E_0}{N} = \frac{3}{5} \left[\frac{\hbar^2 p_f^2}{2m} \right] = \frac{3}{5} \varepsilon_F$$

where $\varepsilon_F = p_f^2/2m$ is called "Fermi Energy".

"Fermi- Pressure"

Recall: $dE = TdS - PdV + \mu dN$

$\Rightarrow$ at $T=0$, $P = -\frac{d}{dV} \langle \varphi_0 | \hat{H} | \varphi_0 \rangle \Big|_N$

$$= -\frac{dE_0}{dV}\Big|_N$$

Using the expression above,

$$E_0 = \frac{3}{5} N \varepsilon_F = \frac{3}{5} N \left[\frac{3\pi^2 N}{V}\right]^{2/3}$$

$$= \frac{3}{5}(3\pi^2)^{2/3} \frac{N^{5/3}}{V^{2/3}}$$

$\Rightarrow$

$$P = \frac{3}{5}(3\pi^2)^{2/3} \frac{N^{5/3}}{2m} \times \frac{2}{3} V^{-5/3}$$

$$= \frac{2}{3}\frac{E}{V}$$

Thus, a fermi gas has a non-zero pressure even at the zero temperature. This is completely different than a classical ideal gas where $P = \rho T = 0$ at $T = 0$.

Correlations in the ground state:

A useful quantity is:

$$G_{\sigma,\sigma'}(r-r') = \langle \psi_0 | a^\dagger(\vec{x},\sigma)\, a(\vec{x}',\sigma') | \psi_0 \rangle$$

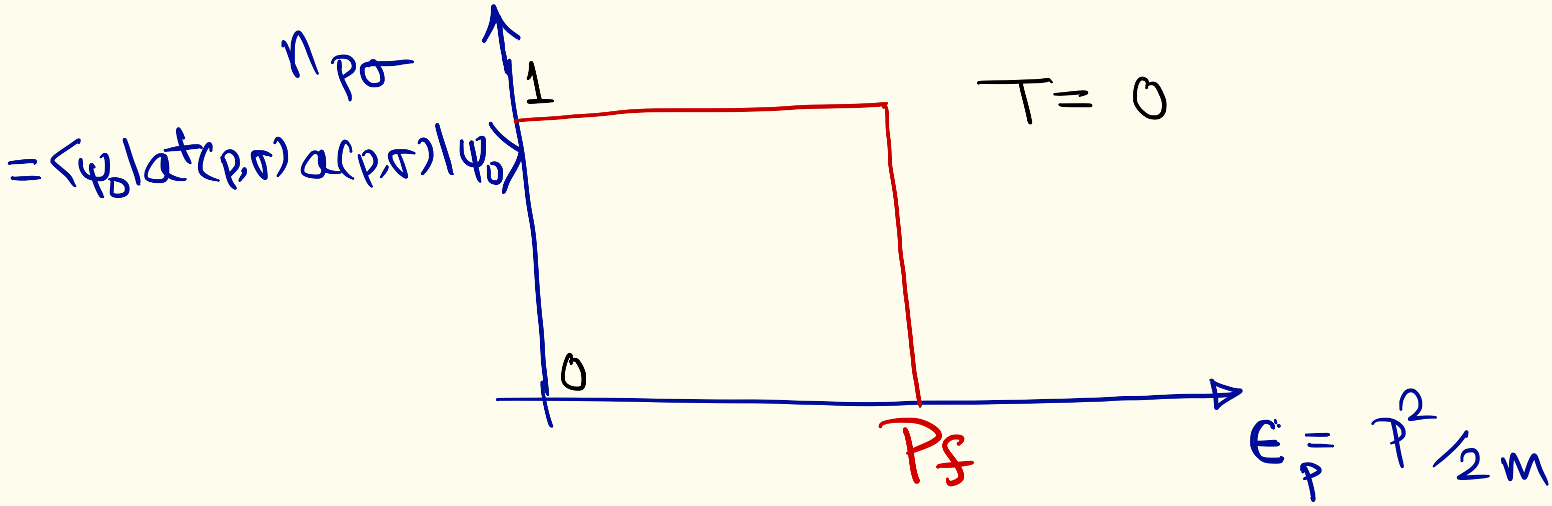
i.e. the amplitude for removing a particle from location $\vec{x}'$ with spin σ' returning to location $\vec{x}$ and spin σ.

Since ground state has a definite value of total σ, $G(r-r', \sigma, \sigma')$ is clearly zero when $\sigma \neq \sigma'$. Thus, we don't need two distinct spin indices:

$$G_\sigma(r-r') = \langle \psi_0 | a^\dagger(\vec{x},\sigma)\, a(\vec{x}',\sigma) | \psi_0 \rangle$$

$$= \frac{1}{V} \sum_{\vec{p}\vec{p}'} \langle \psi_0 | a^\dagger(\vec{p},\sigma)\, a(\vec{p}',\sigma) | \psi_0 \rangle\, e^{i\vec{p}'\cdot\vec{x}' - i\vec{p}\cdot x}$$

$$= \frac{1}{V} \sum_{\vec{p}} \langle \psi_0 | a^\dagger(\vec{p},\sigma)\, a(\vec{p},\sigma) | \psi_0 \rangle\, e^{i\vec{p}\cdot(\vec{x}'-\vec{x})}$$



$$= \frac{1}{V} \sum_{|\vec{p}|<p_f} e^{i\vec{p}\cdot(\vec{x}'-\vec{x})}$$

$$= \frac{V}{8\pi^3}\frac{1}{V}\, 2\pi \int_0^{p_f} p^2\, dp \int_0^{\pi} \sin\theta\, d\theta\; e^{ip|\vec{x}'-\vec{x}|\cos\theta}$$

$$= \frac{1}{4\pi^2} \int_0^{p_f} p^2\, dp \int_{-1}^{1} d\mu \;\; e^{ip|\vec{x}'-\vec{x}|\,\mu}$$

$$= \frac{1}{4\pi^2} \int_0^{p_f} p^2\, dp\, \frac{\left(e^{ip|\vec{x}'-x|} - e^{-ip|x'-x|}\right)}{ip\,|\vec{x}'-\vec{x}|}$$

$$= \frac{1}{4\pi^2} \int_0^{p_f} 2p\, dp\, \frac{\sin(pr)}{r}$$

$$r = |\vec{x}-\vec{x}'|$$

The integral $\int x \sin(\alpha x)\, dx$

$$= -\frac{\cos(\alpha x)\, x}{\alpha} + \int \frac{\cos(\alpha x)}{\alpha}\, dx$$

$$= -\frac{\cos(\alpha x)\, x}{\alpha} + \frac{\sin(\alpha x)}{\alpha^2}$$

$$= \frac{2}{4\pi^2 r}\left[-\frac{\cos(pr)\, p}{r} + \frac{\sin(pr)}{r^2}\right]\Bigg|_0^{p_f}$$

$$= \frac{2}{4\pi^2}\left[-\frac{\cos(p_f r)\, p_f}{r^2} + \frac{\sin(p_f r)}{r^3}\right]$$

define $p_f r = y$

$$= \frac{2 p_f^3}{4\pi^2}\left[\frac{-\cos(y)\, y + \sin(y)}{y^3}\right]$$

Now recall $p_f^3 = 3\pi^2 \rho$

$$\Rightarrow G_r(\vec{r} - \vec{r}') = \frac{3}{2}\rho \left[\frac{\sin(y) - y\cos(y)}{y^3}\right]$$

Thus G shows oscillations at scale p_f^{-1} and decays in a power-law fashion.

The power-law originates from the fact that $\langle n_p \rangle$ is singular in the ground state. The Fourier transform of $G(r-r')$ is also singular at

$$G(\vec{q}) = \frac{1}{\sqrt{V}} \int_{\vec{r}} G(\vec{r})\, e^{+i\vec{q}\cdot\vec{r}}\, d^3r$$

$$\sim \int \frac{d^3r}{\sqrt{V}} \int_{|p|<p_f} d^3p\, e^{-i\vec{p}\cdot\vec{r} + i\vec{q}\cdot\vec{r}}$$

$$\sim \int_{|p|<p_f} \delta(\vec{p}-\vec{q})\, d^3p$$

$$= \begin{cases} 0 & \text{if } |\vec{q}| > p_f \\ 1 & \text{if } |\vec{q}| < p_f \end{cases}$$