Weakly interacting bosons in Continuum

[Reading: Khomskii Chapter 5, Wen Ch. 3, Girvin, Yang Ch. 18]

Review of non-interacting bosons

Consider $H = \sum_k \varepsilon_k b^\dagger_k b_k$ with $\varepsilon_k \sim k^\alpha$

Ground state: $|g.s.\rangle = \left[b^\dagger_{k=0}\right]^N$.

This is a BEC at $T=0$ i.e. $\lim_{N\to\infty} \frac{n_{k=0}}{N} \neq 0$.

Following standard arguments (see 210A notes), as long as BEC exists, chemical potential $\mu = 0$.

For BEC to be stable at $T \neq 0$, the total occupation of $k \neq 0$ single-particle modes must be convergent in the thermodynamic limit.

$\Rightarrow$ $$V \int \frac{d^d k}{e^{\beta \varepsilon_k} - 1}$$ must converge.

If $\varepsilon_k \sim k^\alpha$ $\Rightarrow$ $\boxed{d > \alpha}$.

Order parameter for BEC:

Consider the correlation f^n

$$\langle b^+(r)\, b(r')\rangle = \frac{\mathrm{Tr}\, e^{-\beta H}\, b^+(r)\, b(r')}{\mathrm{Tr}\, e^{-\beta H}}$$

Since the Hamiltonian is translationally invariant, it makes sense to evaluate this by fourier transform. $b(r) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} b(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$

$$\langle b^+(r)\, b(r')\rangle = \frac{1}{V}\sum_{kk'}\langle b^+(k)\, b(k')\rangle\, e^{ik'\cdot r' - ik\cdot r}$$

$$= \frac{1}{V}\sum_k \langle b^+(k)\, b(k)\rangle\, e^{ik\cdot(r-r')}$$

$$= \frac{\langle b^+(k=0)\, b(k=0)\rangle}{V} + \frac{1}{(2\pi)^d}\int \frac{d^dk\, e^{ik\cdot(r-r')}}{e^{\beta\varepsilon_k} - 1}$$

Assuming, $\varepsilon_k \sim k^\alpha$ with $\alpha < d$, so that BEC exists at $T \neq 0$, $\langle b^+(k=0)\, b(k=0)\rangle = N_0 \propto V$

$$= \frac{N_0}{V} + \frac{1}{(2\pi)^d}\int \frac{d^dk\, e^{ik\cdot(r-r')}}{e^{\beta k^\alpha} - 1}$$

$$\sim \frac{N_0}{V} + \frac{1}{\beta} \frac{1}{|r-r'|^{d-\alpha}}$$

$$\neq 0 \quad \text{as} \quad |r-r'| \to \infty .$$

This is a rather striking result since the correlation f^n doesn't decay with distance.

This signals spontaneous symmetry breaking with order parameter $= \langle b(r) \rangle \approx \sqrt{\frac{N_0}{V}}$

Another way to see this is,

$$\langle b(r) \rangle \sim \frac{1}{\sqrt{V}} \langle b(k=0) \rangle \sim \frac{\sqrt{N_0}}{\sqrt{V}}$$

Thus, $b(k=0)$ almost behaves like a c-number instead of an operator: $b(k=0) \sim \sqrt{N_0}$

Weakly interacting BEC = Superfluid

Consider point-like interacting bosons:

$$H = \underbrace{\sum_k \varepsilon_k b^\dagger_k b_k}_{H_0} + \underbrace{\frac{U}{2} \sum_r b^\dagger(r)\, b^\dagger(r)\, b(r)\, b(r)}_{H_{int}}$$

$$= \sum_k \varepsilon_k b^\dagger_k b_k + \frac{U}{V} \sum_{k_1 k_2 k_3} b^\dagger(k_1)\, b^\dagger(k_2)\, b(k_3)\, b(k_1+k_2-k_3)$$

Since $\langle b^\dagger(k=0)\rangle \propto \sqrt{N}$ where N is the # of particles, while $\langle b^\dagger(k\neq 0)\rangle = 0$, in the above sum, the terms involving $b(k=0)$ scale differently than those involving $b(k\neq 0)$. Therefore, we need to separate $b(k=0)$ terms out.

When U is small, to a good approximation at $T=0$. $b(k=0) = \sqrt{N} = b^\dagger(k=0)$

$$\Rightarrow H_{int} = \frac{U}{2V}\Big[(b^\dagger(k=0))^2 (b(k=0))^2 + [b(k=0)]^2 \sum_{k\neq 0}\big[4\, b^\dagger_k b_k + b_k b_{-k} + b^\dagger_k b^\dagger_{-k}\big]\Big]$$

[Note that All of these terms scale as N^2]

The term $[b(k=0)]^4$ requires special care. One might think that it's simply N^2. However, it deviates a little bit from N^2.

$$N = b^2(k=0) + \sum_{k\neq 0} b^+_k b_k$$

$$\Rightarrow \quad b^4(k=0) = \Big[N - \sum_{k\neq 0} b^+_k b_k\Big]^2$$

$$= N^2 - \underbrace{2N \sum_{k\neq 0} b^+_k b_k}_{O(\delta N)} + \underbrace{\sum_{k,k'\neq 0} \big[b^+_k b_k b^+_{k'} b_{k'}\big]}_{O((\delta N)^2)}$$

where $\delta N = N - b^2(k=0)$

Keeping terms to the leading order in δN,

$$H_{int} = \frac{UN^2}{2V} + \frac{UN}{2V} \sum_{k\neq 0} \big[2 b^+_k b_k + b^+_k b^+_{-k} + b_k b_{-k}\big]$$

Combining everything,

$$H = \frac{UN^2}{2V} + \sum_{k\neq 0} \Big[\varepsilon_k + \frac{UN}{V}\Big] b^+_k b_k + \frac{UN}{2V}\sum_{k\neq 0} \big[b^+_k b^+_{-k} + h.c.\big]$$

$$= \frac{UN^2}{2V} + \frac{1}{2} \sum_{k\neq 0} \Big\{\Big[\varepsilon_k + \frac{UN}{V}\Big]\big[b^+_k b_k + b^+_{-k} b_{-k}\big] + \frac{UN}{V}\big(b^+_k b^+_{-k} + h.c.\big)\Big\}$$

This Hamiltonian can be diagonalized via the following transformation:

$$b_k = \cosh(\theta_p)\, a_k + \sinh(\theta_k)\, a^\dagger_{-k}$$

We choose θ_k so that H in terms of $a_k, a^\dagger_k$ has the form $\sum_{k\neq 0} E_k\, a^\dagger_k a_k$ upto constant terms, i.e. the terms such as $a^\dagger_k\, a^\dagger_{-k}$ cancel out.

To keep things slightly more general, consider

$$H = \sum_{p\neq 0} \Big\{ D_p \left[b^\dagger_p b_p + b^\dagger_{-p} b_{-p} \right] + \Delta_p \left[b^\dagger_p\, b^\dagger_{-p} + h.c. \right] \Big\}$$

Substituting b_p in terms of a_p:

$$H = \sum_p 2D_p \left[\cosh^2\theta_p\, a^\dagger_p a_p + \sinh^2\theta_p\, a_{-p} a^\dagger_{-p} + \cosh(\theta_p)\sinh(\theta_p)\left(a^\dagger_p\, a^\dagger_{-p} + h.c.\right)\right]$$

$$+ \sum_p \Delta_p \left[\left(\cosh^2\theta_p + \sinh^2\theta_p\right)\left(a^\dagger_p\, a^\dagger_{-p} + h.c.\right) + 2\cosh\theta_p \sinh\theta_p \left(a^\dagger_p a_p + a_{-p} a^\dagger_{-p}\right)\right]$$

Requiring that the coefficient of $(a^+_p a^+_{-p} + h.c.)$ vanishes $\Rightarrow$ $D_p \sinh(2\theta_p) + \Delta_p \cosh(2\theta_p) = 0$

$$\Rightarrow \quad \tanh(2\theta_p) = -\frac{\Delta_p}{D_p}$$

The Hamiltonian thus becomes (upto a constant)

$$H = \sum_p \Big\{ 2D_p \left[\cosh^2\theta_p + \sinh^2\theta_p\right] + 4\Delta_p \cosh\theta_p \sinh\theta_p \Big] a^+_p a_p$$

$$= \sum_p E_p \, a^+_p a_p$$

where $E_p = 2D_p \cosh(2\theta_p) + 2\Delta_p \sinh(2\theta_p)$

$$= \frac{2D_p}{\sqrt{1-\tanh^2 2\theta_p}} + \frac{2\Delta_p \tanh(2\theta_p)}{\sqrt{1-\tanh^2(2\theta_p)}}$$

$$= 2\sqrt{D_p^2 - \Delta_p^2}\,.$$

Returning to our specific problem,

$$D_p = \frac{1}{2}\left(\varepsilon_p + \frac{UN}{V}\right), \quad \Delta_p = \frac{UN}{2V}$$

$$\Rightarrow E_p = \sqrt{\varepsilon_p^2 + 2\varepsilon_p \frac{UN}{V}}$$

Choosing $\varepsilon_p = \dfrac{p^2}{2m}$, as $p \to 0$, $E_p \approx p\sqrt{\dfrac{UN}{Vm}}$

i.e. $E_p \sim p$ similar to sound waves/phonons.

On the other hand, at high energies, $E_p \sim \varepsilon_p$.

Thus, the low energy effective Hamiltonian is

$$H_{eff} \sim \sum_p p \sqrt{\frac{UN}{Vm}}\, a^\dagger_p a_p$$

This is a superfluid.

Ground state is defined by $a_p |0\rangle = 0$.

Goldstone theorem:

The emergence of a linearly dispersing mode is not a coincidence, but is a consequence of the fact that the system has a broken a continuous symmetry spontaneously.

Proof (Sketch): Consider a system with a conserved charge Q i.e. $[H, Q] = 0$.

Let's assume that the ground state is not invariant under the corresponding symmetry transformation i.e. $e^{i\theta Q}|g.s.\rangle \neq |g.s.\rangle$

$$\Rightarrow \quad Q|g.s.\rangle \neq |g.s.\rangle$$

for example, in our example of the superfluid, $Q = \sum_p b^+{}_p b_p$.

the ground state we obtained, as defined by $a_p|0\rangle = 0$ is not an ~~eigenstate~~ eigenstate of Q, therefore $Q|g.s.\rangle \neq |g.s.\rangle$.

Let's consider the state $Q|g.s.\rangle$.

$$H(Q|g.s.\rangle) = Q H|g.s.\rangle$$
$$= E_0 Q|g.s.\rangle$$

where E_0 is the ground state energy. Therefore $Q|g.s.\rangle$ despite being not same $|g.s.\rangle$, has the same energy. This seems to indicate that there exist low energy modes. To construct a family of these modes, write $Q = \int d^dx\, J_0(x)$ and consider

$$|k\rangle = \int d^dx\, e^{-i\vec{k}\cdot\vec{x}} J_0(x)|g.s.\rangle$$

where $\vec{k}=0$, one recovers $Q|g.s.\rangle$. for small $|\vec{k}|$, one obtains a state whose energy is at most $|\vec{k}|$. These are Goldstone modes.

Ground state energy and compressibility:

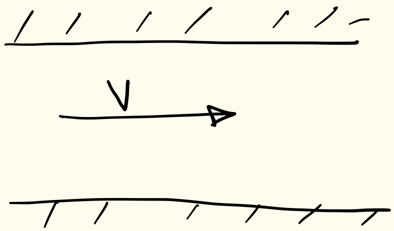
One can show that the ground state energy is $\frac{U}{2V} N(N-1)$. Therefore, the chemical potential $\mu = \left.\frac{\partial E}{\partial N}\right|_V = \left(N - \frac{1}{2}\right)\frac{U}{V}$

$\approx U\rho$ where $\rho = \frac{N}{V}$ is the density.

The compressibility $\kappa = \frac{\partial \rho}{\rho^2 \partial \mu} \sim \frac{1}{\rho^2 U}$,

which is finite, unlike BEC, for which $\kappa = \infty$.

Why Superfluidity?



Consider a moving condensate. To degrade the superflow, one would need to create excitation(s) in the moving frame. Let's assume one such excitation exist with momentum $\vec{p}$ and energy $\varepsilon(\vec{p})$. In the lab frame, the total energy is,

$$E = \frac{1}{2} M \vec{v}^{\,2} + \varepsilon(\vec{p}) + \vec{p}\cdot\vec{v}$$

For the excitation to be energetically favorable, one requires that $\varepsilon(p) + \vec{p}\cdot\vec{v} < 0$

$\varepsilon(p) = c\,|\vec{p}| \Rightarrow \boxed{v > c}$. Thus at slow enough velocities, the superflow is stable. More precisely, $v > \min\left(\frac{\varepsilon(p)}{p}\right)$