But now one notices that <u>neither</u> $|\psi'_1\rangle$ and $|\psi'_2\rangle$ break the Ising symmetry spontaneously. Indeed, upto a phase, both $|\psi'_1\rangle$ and $|\psi'_2\rangle$ are invariant under the Ising symmetry.

$$U|\psi'_1\rangle = |\psi'_1\rangle$$

$$U|\psi'_2\rangle = -|\psi'_2\rangle.$$

So does one conclude that the symmetry is not broken spontaneously when $h \neq 0$?

No! This is a wrong argument!

Recall that to talk about spontaneous symmetry breaking, one has to include an infinitesimal symmetry breaking field $\Delta H = \epsilon \sum_i \sigma_i^z$ and consider the limit $L \to \infty$ first and then $\epsilon \to 0$.

when one introduces an infinitesimal longitudinal field $\epsilon \sum_i \sigma_i^z$, one obtains back the eigenstates $|\psi_1\rangle = \prod_i |\uparrow\rangle_i$ and $|\psi_2\rangle = \prod_i |\downarrow\rangle_i$, thus leading to spontaneous symmetry breaking. The transverse field $h \sum_i \sigma_i^x$ will modify $|\psi_1\rangle$ and $|\psi_2\rangle$ little bit (e.g. one of the ground states will look like

$$|\uparrow\uparrow\uparrow \ldots \uparrow\rangle + \frac{h}{J}\left[|\downarrow\uparrow\uparrow\uparrow \ldots \uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow \ldots \uparrow\rangle + \ldots\right]$$

) but overall, the two low-lying states will be degenerate in the thermodynamic limit and will almost look like $\prod_i |\uparrow\rangle_i$ and $\prod_i |\downarrow\rangle_i$ in the presence of infinitesimal longitudinal field $\epsilon \sum_i \sigma_i^z$.

Therefore, the spontaneous symmetry breaking occurs in the limit $h \ll J$.

Next, consider the limit $h \gg J$.

② $h \gg J$.

Let's first set $J = 0$. There is a unique ground state.

$$|\psi\rangle = \prod_i |\rightarrow\rangle_i \quad \text{where}$$

$|\rightarrow\rangle$ denotes a spin pointing along the $+x$ direction. This unique ground state is invariant under the unitary transformation $U = \prod_i \sigma_i^x$

$$\prod_i \sigma_i^x \prod_j |\rightarrow\rangle_j = \prod_j |\rightarrow\rangle_j$$

Therefore, symmetry is <u>unbroken</u> in the limit $h \gg J$.

Excitations around the two limits :

When $J \gg h$, the excitations are domain walls and cost energy $O(J)$.

$$|\uparrow\uparrow\uparrow\cdots\underbrace{\uparrow\uparrow\;\;\downarrow}_{\text{domain wall}}\downarrow\downarrow\downarrow\cdots\rangle$$

Domain wall is a 'topological defect', such topological defects are a hallmark of spontaneous symmetry breaking.

On the other hand, when $h \gg J$, the excitations are flipped spins from $+x$ direction to $-x$ direction and cost energy $O(h)$.

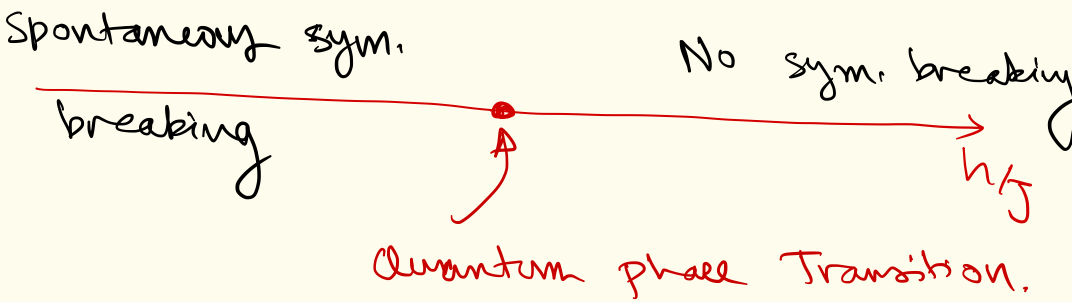
$$|\rightarrow\rightarrow\rightarrow\rightarrow\underset{\text{flipped spin.}}{\leftarrow}\rightarrow\rightarrow\rightarrow\rightarrow\rangle$$

This is <u>not</u> a topological defect.

Lets make a table that summarizes the differences between $h \gg J$ limit and $J \gg h$ limit.

$J \gg h$	$h \gg J$
1. Two degenerate ground states in the thermodynamic limit.	Unique ground state in the thermodynamic limit.
2. Ground states break the Ising symmetry spontaneously.	Ground state does <u>not</u> break the Ising sym. spontaneously.
3. Excitations are domain walls.	Excitations are individually flipped spins

Following our earlier discussion on phases and phase transitions, these two limits correspond to two different phases of matter and therefore must be separated by a phase transition where the ground state energy is a non-analytic function of the tuning parameter h/J



We next study this phase transition using <u>mean-field theory</u>.

Mean field Theory of Quantum Phase Transition in the 1d Transverse field Ising Model.

The actual Hamiltonian is:

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$$

Within the mean-field theory, we replace the interaction term (i.e. the J term) as:

$$\sigma_i^z \sigma_{i+1}^z \rightarrow \langle \sigma_i^z \rangle \sigma_{i+1}^z + \sigma_i^z \langle \sigma_{i+1}^z \rangle - \langle \sigma_i^z \rangle \langle \sigma_{i+1}^z \rangle$$

Further we assume a <u>homogeneous</u> solution $\langle \sigma_i^z \rangle = m \quad \forall i$.

The mean-field Hamiltonian becomes

$$H_{MF} = -2Jm\sum_i \sigma_i^z - h\sum_i \sigma_i^x$$
$$+ NJm^2$$

Were the coordination no. z (here $z=2$), we would have

$$H_{MF} = -zJm\sum_i \sigma_i^z - h\sum_i \sigma_i^x$$
$$+ \frac{Nz}{2}Jm^2$$

The ground state energy is

$$\frac{E}{N} = -\sqrt{(zJm)^2 + h^2} + \frac{Jzm^2}{2}$$

Doing a taylor expansion in m, one obtains,

$$\frac{E}{N} = -h\left[1+\left(\frac{zJm}{h}\right)^2\right]^{1/2} + \frac{Jzm^2}{2}$$

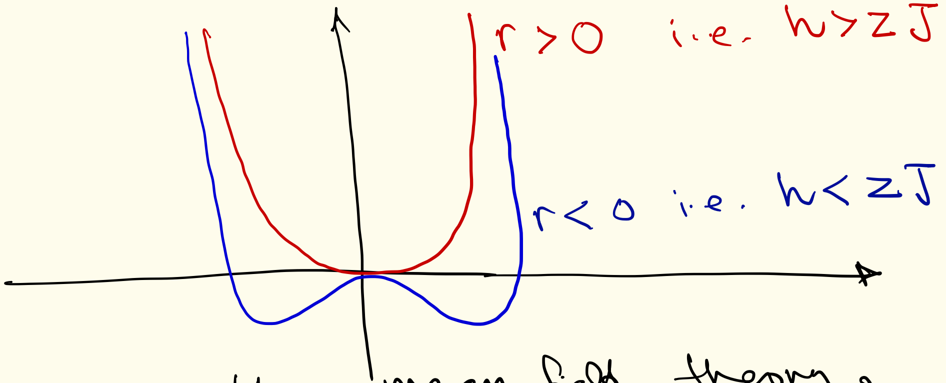
$$\simeq -h\left[1+\frac{1}{2}\frac{(zJm)^2}{h^2} - \frac{1}{4}\left(\frac{zJm}{h}\right)^4 + \cdots\right] + \frac{Jzm^2}{2}$$

$$= r\,m^2 + u\,m^4 + \text{constant}$$

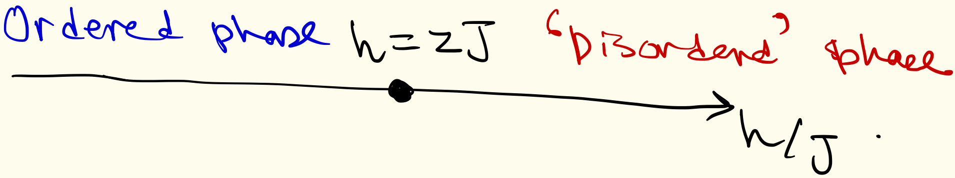
where $r = \dfrac{Jz}{2} - \dfrac{(zJ)^2}{2h}$

$$u = \frac{1}{4}\frac{(zJm)^4}{h^3}$$

This has exactly the same form as the Landau free energy for the Ising transition!



Thus, within the mean-field theory, the sym. is spontaneously broken when $h < zJ$.



An alternate method is to find m self consistently,

$$m = \langle \varphi_0 | \sigma_i^z | \varphi_0 \rangle$$ where

$|\varphi_0\rangle$ is the mean-field ground state.

$$\langle \psi_0 | \sigma_i^z | \psi_0 \rangle = \frac{zJm}{\sqrt{(zJm)^2 + h^2}}$$

$$\Rightarrow \quad m = \frac{zJm}{\sqrt{(zJm)^2 + h^2}}$$

$$\Rightarrow (zJm)^2 + h^2 = (zJ)^2$$

$$\Rightarrow \quad m^2 = \frac{(zJ)^2 - h^2}{(zJ)^2}$$

$$\Rightarrow \quad m = \pm \sqrt{1 - \frac{h^2}{(zJ)^2}}$$

Thus, m vanishes above the critical ratio: $\left(\frac{h}{J}\right)_c = z$, as also seen above using the Landau theory.