Spatial correlations within mean-field theory

In the Landau theory, we let order-parameter to be uniform in space, which is a rather drastic assumption. To improve upon it, let the order-parameter be a function of space, $m(\vec{r})$, and consider the Landau free energy as a functional of $m(\vec{r})$:

$$F = \int d^d r \left[(\nabla m)^2 + t m^2 + u m^4 - h m \right]$$

where $F = f V$ is the extensive Landau free energy, $t = a(T - T_c)$, and $h(r)$ is the magnetic field, which we also allow to be inhomogeneous.

The partition function is given by,

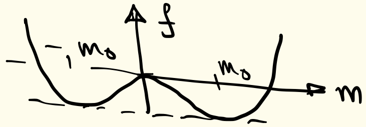
$$Z = \int Dm(r) \, e^{-\beta \int d^d r \left[(\nabla m)^2 + t m^2 + u m^4 - h m \right]}$$

where $\int Dm$ denotes functional integral. Doing this integral exactly is quite hard, but one evaluate it within <u>Gaussian</u> approximation.

For $t > 0$ i.e. $T > T_c$, $\langle m \rangle = 0$, therefore,

$$(\nabla m)^2 + t m^2 + u m^4 \simeq (\nabla m)^2 + t m^2$$

For $t < 0$, $\langle m \rangle \neq 0$ and one can again expand around $\langle m \rangle$ to obtain a gaussian approximation:



Within Landau theory, $\langle m \rangle^2 = -\dfrac{t}{2u}$

write. $m = \langle m \rangle + \delta m$,

$$t m^2 + u m^4 \cong t \langle m \rangle^2 + u \langle m \rangle^4 + \left[2t + 12 u \langle m \rangle^2\right] (\delta m)^2$$

$$= -\frac{t^2}{2u} + \frac{u t^2}{4u^2} + \left[2t + 12u \frac{-t}{2u}\right] (\delta m)^2$$

$$= \frac{-t^2}{4u} + 4|t| (\delta m)^2 .$$

Let's use this Gaussian approx. to calculate correlation f^{n}s above and below T_c; at $h = 0$.

$$\langle m(\vec{r})\, m(\vec{r}\,') \rangle = \frac{1}{V} \int_k \langle |m_k|^2 \rangle e^{ik\cdot(r - r')}$$

where m_k is the Fourier transform of $m(\vec{r})$

Within Gaussian approximation, the partition f^n is:

$t > 0$:

$$Z = \int \mathcal{D}m(r)\, e^{-\beta \int d^d r\, [(\nabla m)^2 + t m^2]}$$

$$= \int \underset{\mathcal{D}m^*_k}{\mathcal{D}m_k}\, e^{-\beta \int d^d k\, |m_k|^2 [t + k^2]}$$

$$\Rightarrow \langle |m_k|^2 \rangle \sim \frac{T}{t + k^2}$$

$$\Rightarrow \langle m(r)\, m(r') \rangle \sim \frac{1}{V} \int_k \frac{T}{[t + k^2]}\, e^{i k \cdot (\vec{r} - \vec{r}')}\, d^d k$$

$$\sim T\, \frac{e^{-|r - r'|/\xi}}{|\vec{r} - \vec{r}'|^{d-2}}$$

where $\xi \sim \frac{1}{\sqrt{t}}$.

Thus, as $r \to 0$ (i.e. as $T \to T_c^+$), the correlation length diverges. This is a hallmark of a second-order transition.

$t < 0$: Here $Z = \int \mathcal{D}\, \delta m(r)\, e^{-\beta \int d^d r\, [[\nabla \delta m]^2 + 4|t|(\delta m)^2]}$

Thus, using the above result

$$\langle \delta m(r)\, \delta m(r') \rangle \sim \frac{T\, e^{-|r - r'|/\xi}}{|\vec{r} - \vec{r}'|^{d-2}}$$

i.e. $\langle [m(r) - \langle m \rangle][m(r') - \langle m \rangle] \rangle$

$$= \langle m(r)\, m(r') \rangle - \langle m \rangle^2$$

$$\sim T \frac{e^{-|\vec{r}-\vec{r}'|/\xi}}{|\vec{r}-\vec{r}'|^{d-2}} \quad \text{where} \quad \xi \sim \frac{1}{\sqrt{|t|}}$$

Thus again, as $t \to 0$ (i.e. as $T \to T_c^-$), the correlation length diverges.

Critical exponents η, ν:

Going beyond mean-field, the actual correlation functions behave as:

$$\langle m(r)\, m(r') \rangle \sim \frac{e^{-|\vec{r}-\vec{r}'|/\xi}}{|\vec{r}-\vec{r}'|^{d-2+\eta}}$$

where $\xi \sim \dfrac{1}{|t|^{\nu}} \sim \dfrac{1}{(T-T_c)^{\nu}}$.

Thus, within mean-field, $\eta = 0$ and $\nu = \dfrac{1}{2}$.

Quantum Phase Transitions

Quantum fluctuations provide an alternative route to go from one phase to another. This is most manifest at zero kelvin when there are no thermal fluctuations. Interestingly, as earlier discussed in brief, one can map a quantum system in d-space dimensions at zero temperature to a classical system in d+1-space dimensions at non-zero temperatures. This mapping therefore translates quantum fluctuations in d-dimensions to classical fluctuations in d+1-dimensions.

One of the simplest example of a quantum phase transition is provided by an Ising model in $d=1$ in the presence of a transverse magnetic field, to which we now turn our attention.

Transerve Field Ising Model in d = 1

The Hamiltonian is given by

$$\hat{H} = -J\sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$$

Ising Coupling

Transvrerse field

When $h=0$, this is identical to 1d classical Ising model, which, as you recall, can be solved using Transfer matrix method.

Also, it's crucial to note that the above model is very different from the 'longitudinal field Ising model',

$$\hat{H} = -J\sum_i \sigma_i^z \sigma_{i+1}^z - h\sum_i \sigma_i^z$$ which

doesn't have the $\sigma_i^z \to -\sigma_i^z$ symmetry (we will refer to this symmetry as 'Ising symmetry' as is conventional).

Symmetry of H:

The Ising symmetry of the transverse field Ising model is a consequence of the fact that

$$U^\dagger H U = H$$

where $U = \prod_i \sigma_i^x$

Note that $U^\dagger U = 1$ and $U^2 = 1$

Also $U^\dagger \sigma_i^z U = -\sigma_i^z$ and therefore if we add a longitudinal field $\sum_i \sigma_i^z$ to H, it will <u>lose</u> the Ising symmetry.

Since symmetries can be spontaneously broken, we next ask: does the ground state(s) of H have the Ising symmetry or not?

Spontaneous Symmetry Breaking in the Ground state?

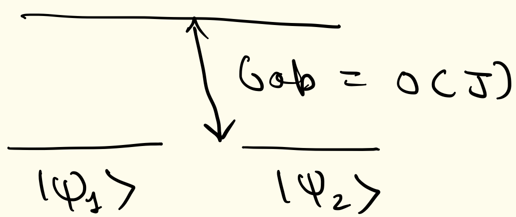
To address this question, we consider the two limits: $J \gg h$ and $J \ll h$.

① $J \gg h$

This is the familiar limit because in this limit the ground state(s) essentially behave like the classical 1d Ising model at $T=0$ $(h=0)$. At $h=0$, there are two ground states:

$$|\psi_1\rangle = \prod_i |\uparrow\rangle_i \qquad |\psi_2\rangle = \prod_i |\downarrow\rangle_i$$

Therefore at $h=0$, the spectrum looks like:



The two ground states $|\varphi_1\rangle$, $|\varphi_2\rangle$ break the Ising symmetry spontaneously because unlike the Hamiltonian H, $|\varphi_1\rangle$, $|\varphi_2\rangle$ are <u>not</u> unchanged by the action of unitary $U = \prod_i \sigma_i^x$.

Specifically,
$$U|\varphi_1\rangle = |\varphi_2\rangle$$
$$U|\varphi_2\rangle = |\varphi_1\rangle$$

i.e. the two ground states interchange under the action of H.

Therefore, the Ising symmetry is spontaneously broken at $h=0$.

What happens when $h \neq 0$ but still $h \ll J$?

The two ground states will mix with each other at N'th order in perturbation theory in h because one has to flip <u>all</u> spins to go from

$|\varphi_1\rangle$ to $|\varphi_2\rangle$. Therefore, the degenerate perturbation theory will now yield the following two eigenstates,

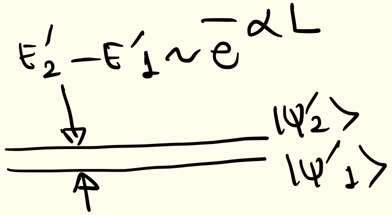
$$|\varphi'_1\rangle = \frac{1}{\sqrt{2}}\left[|\varphi_1\rangle + |\varphi_2\rangle\right]$$

$$|\varphi'_2\rangle = \frac{1}{\sqrt{2}}\left[|\varphi_1\rangle - |\varphi_2\rangle\right]$$

with the eigen-energy difference

$$E'_2 - E'_1 = \frac{h^L}{J^{L-1}} \propto h e^{-\alpha L}$$

where $\alpha \sim \log(J/h)$. Thus, the spectrum looks like



In the thermodynamic limit, $L \to \infty$, the two eigenstates are again degenerate.