$$\hat{S}_x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{S}_y = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$\hat{S}_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where we have chosen to represent the operators in the z-basis so that $\hat{S}_z$ operator is diagonal.

One can also write down classical analogs of the above Hamiltonian.

$$H = -J\sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

where $\vec{S} = [S_x, S_y, S_z]$ is now a three-component vector (instead of operator).

In general, one can write down a spin-model where each spin is an M-component vector:

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \qquad \vec{S}_i = [S_i^1, S_i^2, \ldots S_i^M]$$

$$= -J \sum_{\langle ij \rangle} \sum_{a=1}^{M} S_i^a \, S_j^a$$

For the Ising model, we argued that it has no-ordering in $d=1$ for $T>0$, while it orders in $d>1$.

Is this statement true also for the above M-component model??

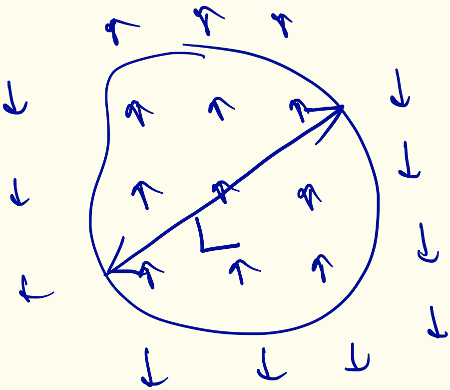
Ans: The above M-component model has no ordering in $d \leq 2$ for $T>0$. In $d=2$, there exists an interesting "power-law ordering" for $M=2$ only.

Domain Wall Arguement

We return to entropy-energy balance arguements based on domain walls, similar to the Ising model.

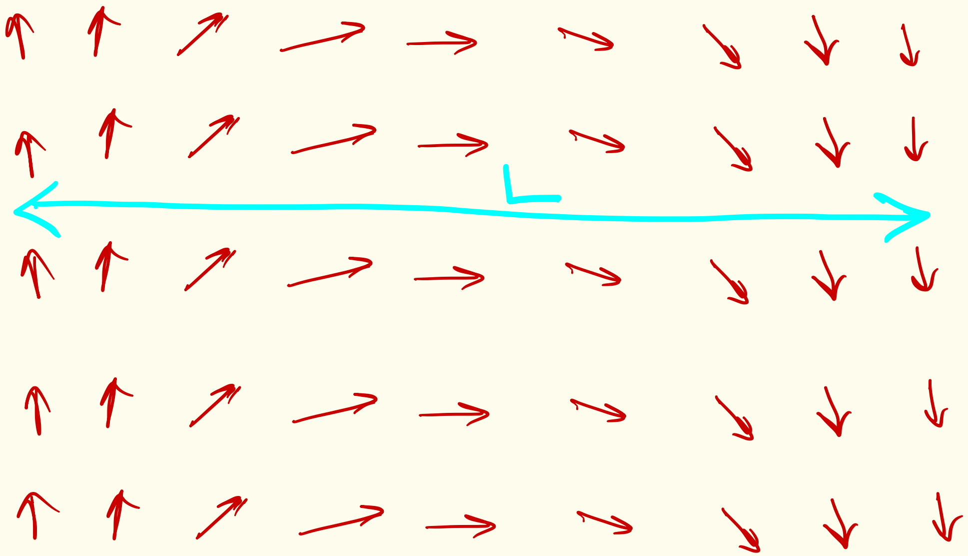
The crucial difference between the Ising model and the above continuous symmetry model is that domain walls cost less energy in $d=2$.

First recall Ising in $d=2$



Energy cost $\sim L$

For the above vector model, one can draw a smaller energy domain wall:



How to estimate energy of above configuration??

$$\sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j = \sum_{\langle ij \rangle} S_i^a \, S_j^a$$

$$\simeq \int (\nabla S_i^a)^2 \, d^d x$$

∇S_i^a for above configuration

$$\simeq \frac{1}{L} \Rightarrow \text{Energy cost} \sim \left(\frac{1}{L}\right)^2 L^d$$

$$\sim L^{d-2}$$

Since entropy is atleast $\log(CL)$

$$\Rightarrow F = L^{d-2} - T\log(CL)$$

in $d=2$, entropy wins

$\Rightarrow$ Always disordered.

in $d=3$ Energy $\sim L$

Entropy can at most scale as $\sim L$

$\Rightarrow$ Ordering possible.

Actual result: in $d=3$, a finite temperature transition between a low-T magnetic phase and high-T non-magnetic phase.

$d=2$ is special because there exists a finite-T transition without an ordered phase at low-T. when $M=2$.

$d = 2\ ,\ M = 2$

The Hamiltonian is

$$H = -J \sum_{\langle ij \rangle} \sum_{a=1,2} S_i^a \, S_j^a$$

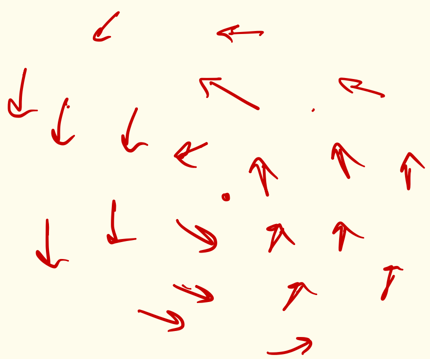
on square lattice (say).

Writing $\vec{S}_i = (\cos\theta_i\ ,\ \sin\theta_i)$

$$\Rightarrow \quad H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

What kinds of excitations exist in this system?

Vortices!



Energy cost of vortex

$$= J \int d^2x \, (\nabla\theta)^2 \qquad \nabla\theta \sim \frac{1}{r}$$

$$J \int_0^L \frac{r\,dr}{r^2} \sim J \log(L)$$

$\neq$ const.

Entropy of a vortex

$= \log(L)$ (because can put it anywhere in the system)

$$F = \log(L)\,[J - T]$$

$\Rightarrow$ Finite temperature phase transition at $T \sim J$.

Low temperature phase: $\vec{m} = \langle \vec{S} \rangle = 0$

but $\langle \vec{m}(r) \cdot \vec{m}(0) \rangle \sim \frac{1}{r^{\eta}}$

and <u>not</u> $e^{-r/\xi}$.

Mean-Field Theory for M-vector Models

This is very similar to the case of Ising spins.

Schematically:

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

$$\Rightarrow H_{\text{mean-field}} = -Jz \sum_{i=1}^{N} \langle \vec{S}_i \rangle \cdot \vec{S}_i$$

Consider the case $M=3$, and chose $\langle \vec{S}_i \rangle = m$ along the z-direction and independent of i

$$\Rightarrow H_{M-F} = -Jmz \sum_{i=1}^{N} S_i^z + \text{const.}$$

$$\Rightarrow Z = \left[\int e^{\beta J m z \cos(\theta_i)} d\theta_i \sin\theta_i \, d\phi_i \right]^N$$

$$\propto \left[\cosh(\beta J m z) \right]^N$$

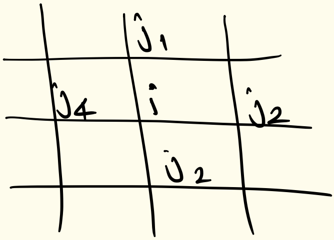
$\Rightarrow$ same result as the Ising model.

Ordering at $T_c = Jmz$.

Clearly, this works only when $d > 2$ as discussed above.

$$\boxed{\text{Improving the Mean-Field Theory (Coupled Cluster MFT)}}$$

One way to think about the MFT is that we replace the interaction of spin i with the rest of the system by an effective magnetic field seen by i.



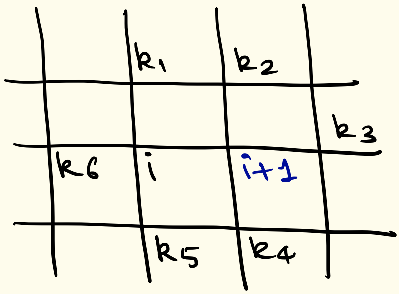
$$S_i S_{j_1} + S_i S_{j_2} + S_i S_{j_3} + S_i S_{j_4}$$

$$\simeq S_i \langle S_{j_1} \rangle + S_i \langle S_{j_2} \rangle + S_i \langle S_{j_3} \rangle + S_i \langle S_{j_4} \rangle + \langle S_i \rangle S_{j_1} + \langle S_i \rangle S_{j_2} + \langle S_i \rangle S_{j_3} + \langle S_i \rangle S_{j_4}$$

[# of terms $= 2N_b = \frac{2zN}{2} = zN$]

$$\Rightarrow H_{M-F} = Jmz \sum_i S_i$$

To improve this, lets consider two spins on which the rest of the system acts like an effective magnetic field.



Effective Hamiltonian for spins i, j in mean-field

$$= -J S_i S_{i+1} - J S_i m \times 3$$

$$- J S_{i+1} m \times 3$$

$$\Rightarrow Z = \left[\sum_{\substack{S_i, S_{i+1} \\ = \pm 1}} e^{\beta J S_i S_{i+1} + 3\beta J m (S_i + S_{i+1})} \right]^{N_b}$$

N_b = # of bonds.

Would the critical temperature go down or up?

Ans: will go down.