First Order Transition in Landau Theory

Consider a system where the Landau free energy is given by:

$$f = \frac{r\, m^2}{2} - w\, m^3 + u\, m^4$$

[$T_2 \neq T_c$. see below]

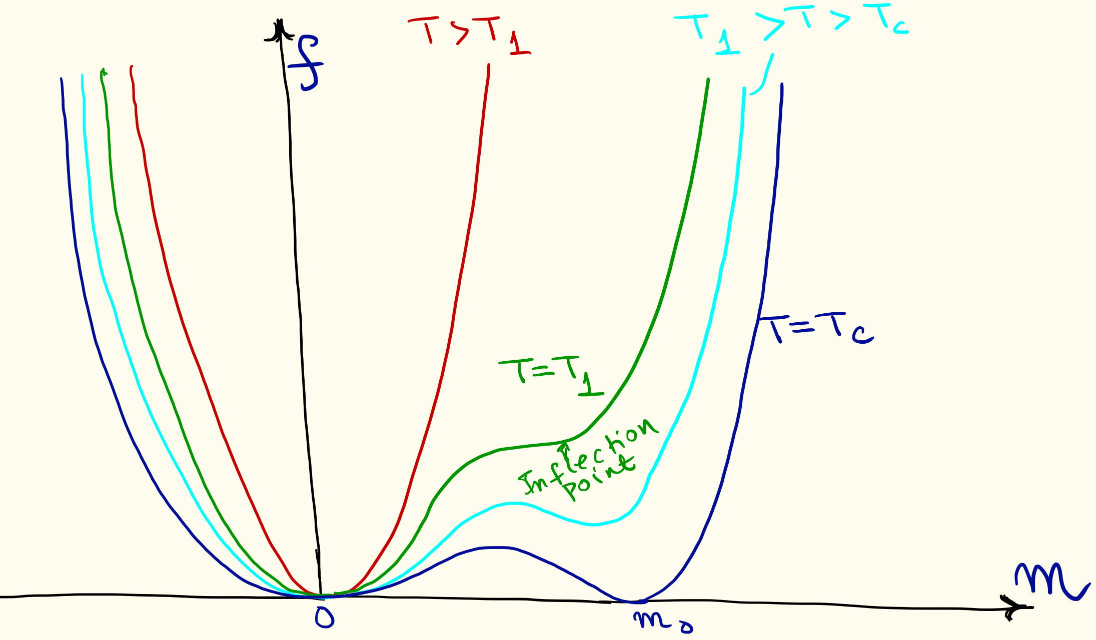
where m is the order parameter, $r = a(T - T_2)$ and u, w are <u>independent</u> of the temperature.

Two physical systems which are described by the above form of free energy are liquid crystals and liquid to solid transition.

Text

What is the phase diagram and the nature of phase transitions?

Let us plot f as a function of temperature for a range of temperatures.



For $T > T_1$, only one local minima.

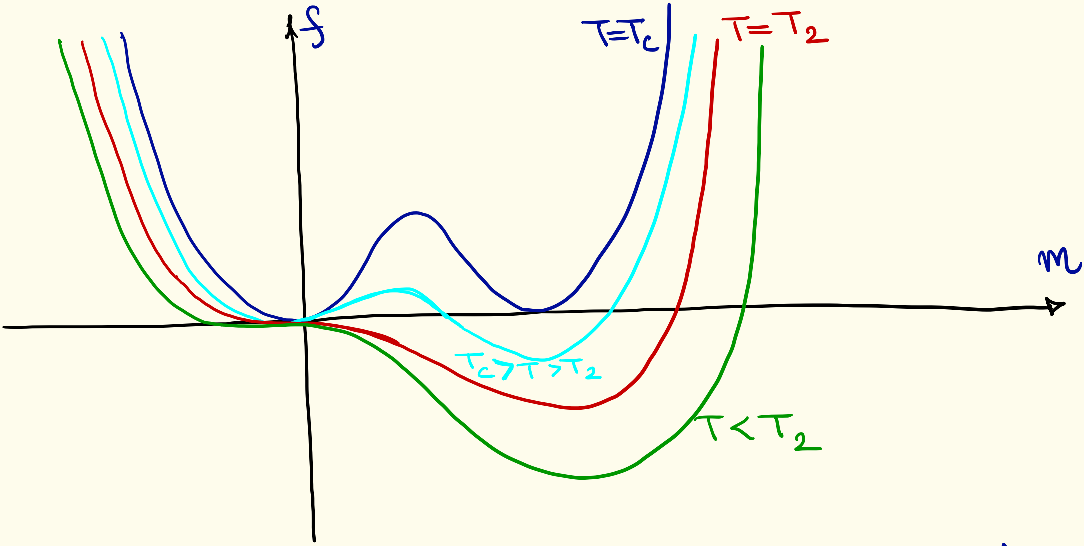
For $T_c > T > T_1$, <u>two</u> local minima, global minima still at $m = 0$.

At $T = T_c$, the two minima are degenerate (i.e. have same free energy f).

For $T > T_c$, the global minima is at $m \neq 0$. Note that m_0 is $O(1)$,

$\Rightarrow$ First-Order transition.

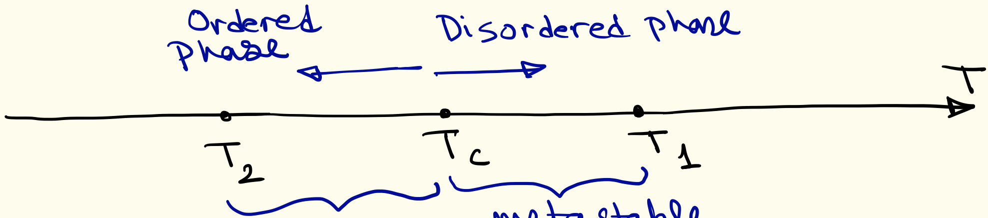
Lets plot f for $T < T_c$ now.



For $T_c > T > T_2$, two local minima, one at $m=0$ and the other at $m \neq 0$. The global minima continues to be at $m \neq 0$.

At $T = T_2$, one looses the local minima at $m = 0 \Rightarrow$ for $T < T_2$, only one minima (at $m \neq 0$).

Phase Diagram as a f^n of Temperature



metastable disordered phase if cooling from $T > T_c$.

metastable ordered phase if heating from $T < T_c$

Let's determine T_c and T_1 in terms of the parameters in the Landau free energy (r, u, T_2, w)

At T_c, $\frac{\partial f}{\partial m} = 0$ has two solutions

$m=0$, and $\underset{(=m_0)}{m \neq 0}$ such that $f(m=0) = f(m_0)$.

$$\frac{\partial f}{\partial m} = (r - 3wm + 4um^2)m = 0$$

$$f = \left[\frac{r}{2} - wm + um^2\right]m^2 = f(m=0) = 0$$

The above eqns. have two solns,

$$m = 0$$

and

$$\left.\begin{aligned} r - 3wm_0 + 4um_0^2 &= 0 \\ \frac{r}{2} - wm_0 + um_0^2 &= 0 \end{aligned}\right\} \text{coupled eqns for } m_0$$

$$\Rightarrow \quad \frac{wm_0}{2} = um_0^2 \Rightarrow m_0 = \frac{w}{2u}$$

$$r = a(T_c - T_2) = \frac{w^2}{2u}$$

$$\Rightarrow \quad \boxed{T_c = T_2 + \frac{w^2}{2ua}}$$

<u>Latent heat</u>: Since the order parameter jumps across the transition, the transition is first order and is accompanied by latent heat.

It is essential to note that the free energy is continuous across the transition (it always is), as is obvious from the plots above.

$$\Delta F = F_{ordered} - F_{disordered}$$
$$= \Delta E - T\,\Delta S = 0$$

$$\Rightarrow \quad \underbrace{\Delta E}_{-\text{Latent heat}} = T_c \underbrace{\Delta S}_{S_{ordered} - S_{disordered}}$$

What is ΔS?

Recall, $S = -\dfrac{\partial F}{\partial T}$

$= -V \dfrac{\partial f}{\partial T}$ (V = volume of the system)

$= -\dfrac{V a m^2}{2}$

$\Rightarrow S_{ordered} - S_{disordered}$

$$= -\frac{V a m_0^2}{2} = -\frac{1}{2} V a \left(\frac{w}{2u}\right)^2$$

$\Rightarrow$ Latent heat

$= -\Delta E = E_{disordered} - E_{ordered}$

$= -T_c \Delta S$

$= \frac{T_c}{2} V a \left(\frac{w}{2u}\right)^2$

$= V\left[T_2 + \frac{w^2}{2ua}\right] a \left(\frac{w}{2u}\right)^2$

Let's calculate T_1 now. At T_1, $\frac{\partial f}{\partial m} = 0$ and $\frac{\partial^2 f}{\partial m^2} = 0$ has a non-zero solution at $m \neq 0$.

$$\frac{\partial^2 f}{\partial m^2} = a(T_1 - T_2) - 6wm_0 + 12um_0^2 = 0$$

$$\frac{\partial f}{\partial m} = [a(T_1 - T_2) - 3wm_0 + 4um_0^2]m = 0$$

$\Rightarrow 3wm_0 = 8um_0^2 \Rightarrow m_0 = \frac{3w}{8u}$

$\Rightarrow a(T_1 - T_2) = 6w\, 3w/8u - 12u \frac{9w^2}{64u^2}$ $\underline{\underline{8u}}$ $9w^2/16u$

$$\Rightarrow \boxed{T_1 = T_2 + \frac{9w^2}{16ua}}$$

Hysterisis, Superheating and Supercooling

Naively, the phase transition occurs at $T=T_c$. But imagine cooling the system from $T>T_c$. The local minima at $m=0$ will make it difficult for the system to 'tunnel' to the global minima until the temperature $T=T_2$ is reached (i.e. when the global minima at $m=0$ is lost). This is the phenomena of 'superheating'. The system can support a non-equilibrium disordered state.

Similarly, lets start from $T < T_2$ and heat the system. The local minima at $m \neq 0$ survives until $T = T_1$, thus, the system will be in a non-equilibrium ordered state for $T_1 > T > T_c$. The actual transition will take somewhere between T_1 and T_c, depending on the dynamics of the system (which we haven't had the chance to discuss).

This is the phenomena of '<u>super cooling</u>'.

Continuous Symmetries

So far we have focussed on the Ising models of various kinds, where spins only take only values $S_i = \pm 1$, or slight generalizations where spins take a few number of discrete values (says $S_i = \pm 1, 0$).

In real systems, there often exist cases where spins take a continuos set of values. For example, electrons carry spin-1/2 and there exist models where the Hamiltonian is:

$$H = -J \sum_{\langle ij \rangle} \vec{\hat{S}}_i \cdot \vec{\hat{S}}_j$$

where $\vec{S} = [\hat{S}_x, \hat{S}_y, \hat{S}_z]$ is a three component operator with