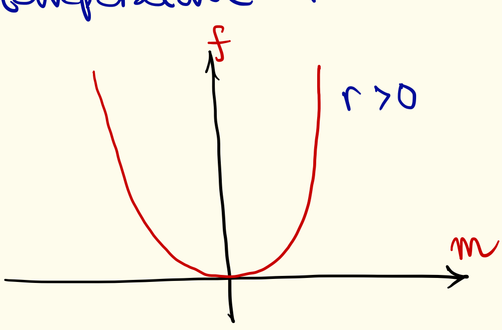
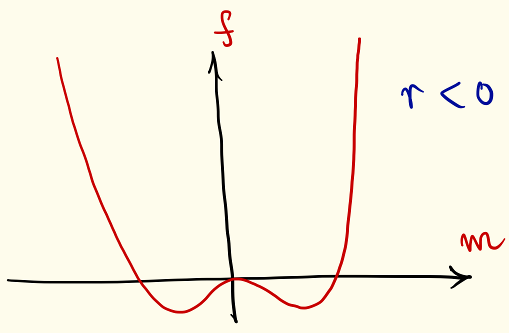
Landau Theory of Phases and Phase Transitions.

Above we saw that the free energy density for the Ising model at $h=0$ takes the following form:

$$f = r m^2 + u m^4$$

where m is the average magnetization and $r \propto T_c - T$ while $u > 0$. Thus, r changes sign across the T_c as the temperature is varied





One could have written down the above expression for $f(m)$ solely based on the symmetry: f must be invariant under $m \leftrightarrow -m$ since at $h=0$, the Hamiltonian is invariant under $S_i \to -S_i$, and since $m = \langle S_i \rangle$, $f(m)$ inherits this symmetry.

For a more general case, the analog of m is called <u>Order parameter</u>.

When the order parameter is <u>zero</u>, the system is in a symmetry <u>preserving</u> phase, and when it is <u>non-zero</u>, the symmetry is <u>spontaneously broken</u>.

Again, we reemphasize that the symmetry is "<u>spontaneously</u>" broken only if the Hamiltonian (and thus, the free energy) has that symmetry to <u>begin with</u>. For example, in the presence of a magnetic field, $f(m)$ will not be symmetric under $m \to -m$, due to the presence of the term hm. In this case, even though $m \neq 0$, the symmetry is not broken spontaneously, but explicity.

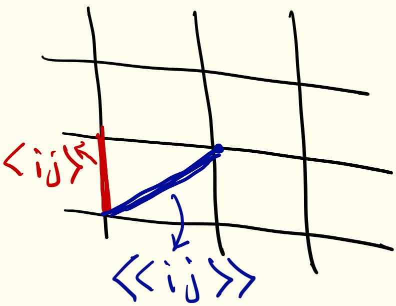
Spontaneous symmetry Breaking (SSB)	Explicit Symmetry Breaking
1. Hamiltonian has the symmetry but the state of the system doesn't 'e.g. Ising model at $h \to 0^+$ at $T < T_c$.	Neither the Hamiltonian, nor the state has the symmetry e.g. Ising model at $h \neq 0$ at any T.
2. Provides an understanding of the exsitence of distinct phazes of matter.	Explains essentially nothing.

The best part of the Landau theory is that it only depends on the symmetries and not the details of the Hamiltonian. This is a very deep fact related to the universality of phases and phase transitions. E.g. consider the Hamiltonian:

$$H = -J_1 \sum_{\langle ij \rangle} S_i S_j - J_2 \sum_{\langle\langle ij \rangle\rangle} S_i S_j$$

$$- h \sum_i S_i$$

where $\langle\langle i,j \rangle\rangle$ denotes next-nearest neighbor exchange on the square lattice.



The Landau free energy is unchanged!

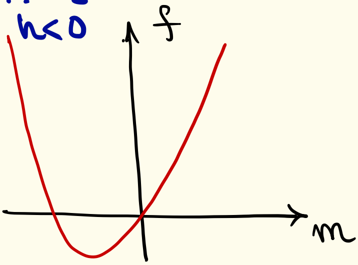
$$f = r m^2 + u m^4 + h m$$

since the symmetry is still the same.

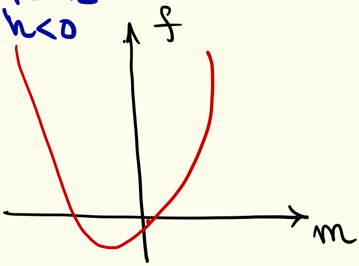
Quick recap of Landau free energy for Ising model: $\left(f = r m^2 + u m^4 - h m\right)$

$r \propto T_c - T$

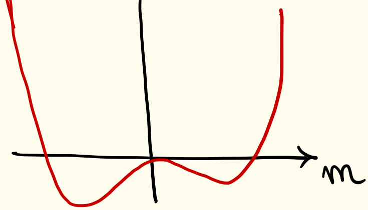
$T > T_c$, $h < 0$



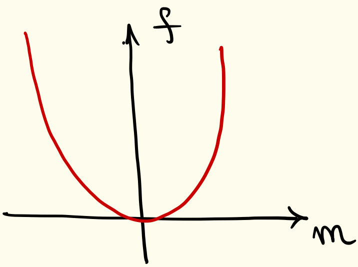
$T = T_c$, $h < 0$



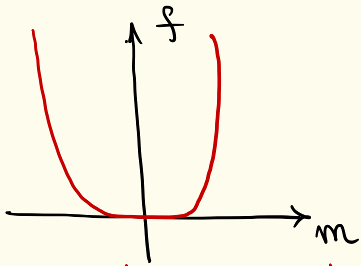
$T < T_c$, $h < 0$



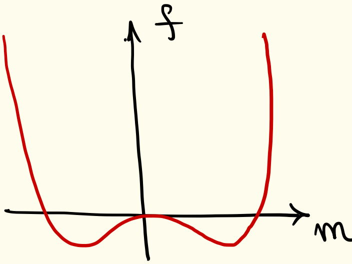
$T > T_c$, $h = 0$



$T = T_c$, $h = 0$

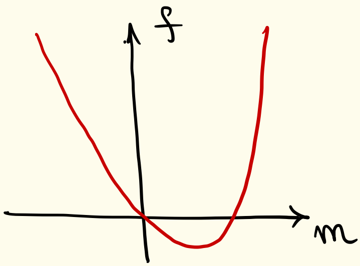


$T < T_c$, $h = 0$

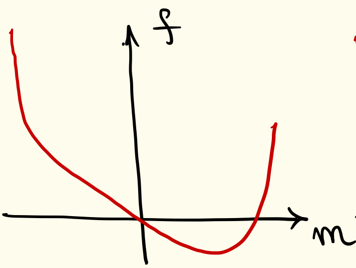


Second-order transition!

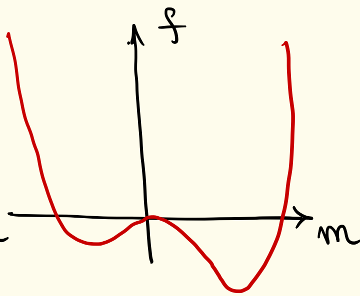
$T > T_c$, $h > 0$



$T = T_c$, $h > 0$



$T < T_c$, $h > 0$



Critical Exponents from Landau Theory

The Landau free energy corresponding to the Ising model is:

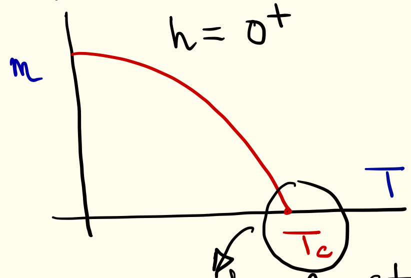
$$f = r m^2 + u m^4 - h m$$

Since r changes sign across T_c, we write it as $r = a(T - T_c)$.

Let's study the critical behavior of this system:

① $\boxed{\text{Dependence of } m \text{ on } T - T_c \text{ at } h = 0}$.

We expect m vs T graph to look like:



What's the analytical structure of this plot close to T_c??

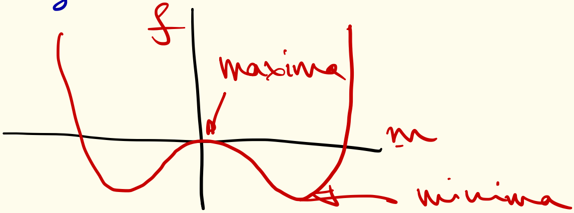
Minimizing f w.r.t. m at $h = 0^+$.

$$\Rightarrow \quad 2mr + 4um^3 = 0$$

Two solutions $m = 0$ and $m^2 = -\frac{r}{2u}$.

The $m^2 = -\frac{r}{2u}$ makes sense only when $r < 0$ i.e. $T < T_c$ because m^2 is positive. $\Rightarrow$ $m = 0$ for $T \geq T_c$.

Further, for $r < 0$, $m^2 = -\frac{r}{2u}$ has a lower free energy compared to $m^2 = 0$ as one may readily check by plugging these two solns into $f(m)$. Or, one may simply note that $\frac{\partial^2 f}{\partial m^2}$ is positive at $m^2 = -\frac{r}{2u}$ $\Rightarrow$ local minima while $\frac{\partial^2 f}{\partial m^2}$ is negative at $m^2 = 0$ $\Rightarrow$ local maxima.



$$\Rightarrow \quad m = \sqrt{\frac{-r}{2u}} \sim (T_c - T)^{1/2}$$

$\Rightarrow$ critical exponent $\beta = 1/2$.

(2) $\boxed{\text{Dependence of } m \text{ on } h \text{ at } T = T_c.}$

At $T = T_c$, $r = 0$.

Minimizing f : $4um^3 - h = 0$

$$\Rightarrow \quad m \sim h^{1/3}$$

$\Rightarrow$ critical exponent $\delta = 3$.

(3) $\boxed{\text{Spin susepetibility at } h = 0^+ \text{ as a } f^n \text{ of } T - T_c.}$

The magnetization m satisfies the equation

$$\frac{\partial f}{\partial m} = 0 \quad \Rightarrow \quad 2mr + 4um^3 - h = 0.$$

Differentiating w.r.t $h \Rightarrow$ $\boxed{2r\frac{\partial m}{\partial h} + 12um^2\frac{\partial m}{\partial h} = 1}$

Now consider two cases, $r > 0$ and $r < 0$ separately, since m is a singular f^n of r.

3a. $r > 0$:

For $r > 0$ i.e. $T > T_c$, $m = 0$

$\Rightarrow$ $2r \dfrac{\partial m}{\partial h} = 1$

$\Rightarrow$ $\left.\dfrac{\partial m}{\partial h}\right|_{h=0^+} = \dfrac{1}{2r} = \dfrac{1}{2|r|}$ $(r>0)$

$\Rightarrow$ $\left.\partial m/\partial h\right|_{h=0^+} \sim \dfrac{1}{|T-T_c|} \Rightarrow \gamma = 1$ exponent

3b. $r < 0$:

For $r < 0$, at $h = 0^+$, we just derived above, $\left.m^2\right|_{h=0^+} = -\dfrac{r}{2u}$

$\Rightarrow$ $(2r + 12u\,m^2)\dfrac{\partial m}{\partial h} = 1$

$\Rightarrow$ $-4r \left.\dfrac{\partial m}{\partial h}\right|_{h=0^+} = 1$

$\Rightarrow$ $\left.\dfrac{\partial m}{\partial h}\right|_{h=0^+} = \dfrac{-1}{4r} = \dfrac{1}{4|r|}$

(since $r<0$)

$\sim \dfrac{1}{|T-T_c|} \Rightarrow \gamma = 1$ exponent

Also, $\boxed{\dfrac{\partial m/\partial h|_{r>0}}{\partial m/\partial h|_{r<0}} = 2}$ $\rightarrow$ Also Universal