Free energy in the Mean-Field Theory

Above we claimed that for $T < T_c$, $m \neq 0$ solution has a lower free energy. Let's show this explicity.

The effective Hamiltonian at $h=0$ is

$$H = -Jmz\sum_i S_i + Jm^2 N_b$$

$$Z = \sum_{\{S\}} e^{+\beta Jmz\sum_i S_i - \beta J m^2 N_b}$$

$$= e^{-\beta J m^2 N_b} \, e^{N\log[2\cosh(J\beta m z)]}$$

$$\Rightarrow F = -T\log Z$$

$$= Jm^2 N_b - N\beta^{-1}\log[2\cosh(J\beta m z)]$$

$$= \frac{Jm^2 zN}{2} - N\beta^{-1}\log[2\cosh(J\beta m z)] \qquad \left(N_b = \frac{zN}{2}\right)$$

First as a check we not that $\frac{\partial F}{\partial m} = 0$ gives

$$JmzN = N\beta^{-1}J\beta z\tanh(J\beta m z)$$

or; $$m = \tanh(J\beta m z)$$

which is exactly the self-consistent eqn

we derived earlier. This is re-assuring.
When 'm' is small (i.e. close to putative T_c), one can Taylor expand F:

$$\frac{F}{N} \cong \frac{Jm^2 z}{2} - T\log 2 - T\log\left(1 + \frac{(\beta J m z)^2}{2!} + \frac{(\beta J m z)^4}{4!} + \dots\right)$$

keeping terms to $O(m^4)$:

$$\frac{F}{N} \cong \frac{Jm^2 z}{2} - T\left[\frac{(\beta J m z)^2}{2!} + \frac{(\beta J m z)^4}{4!} - \frac{(\beta J m z)^4}{8}\right]$$

$$= \frac{Jm^2 z}{2}\left[1 - \beta J z\right] + \frac{\beta^3 (Jz)^4}{12} m^4 - T\log(2)$$

This is a very interesting result!

① $f = \frac{F}{N}$ is symmetric under $m \to -m$ as it should be: since $h=0$, system has $s \to -s$ symmetry. And since $m = \langle s \rangle$, $m \to -m$ sym. for F.

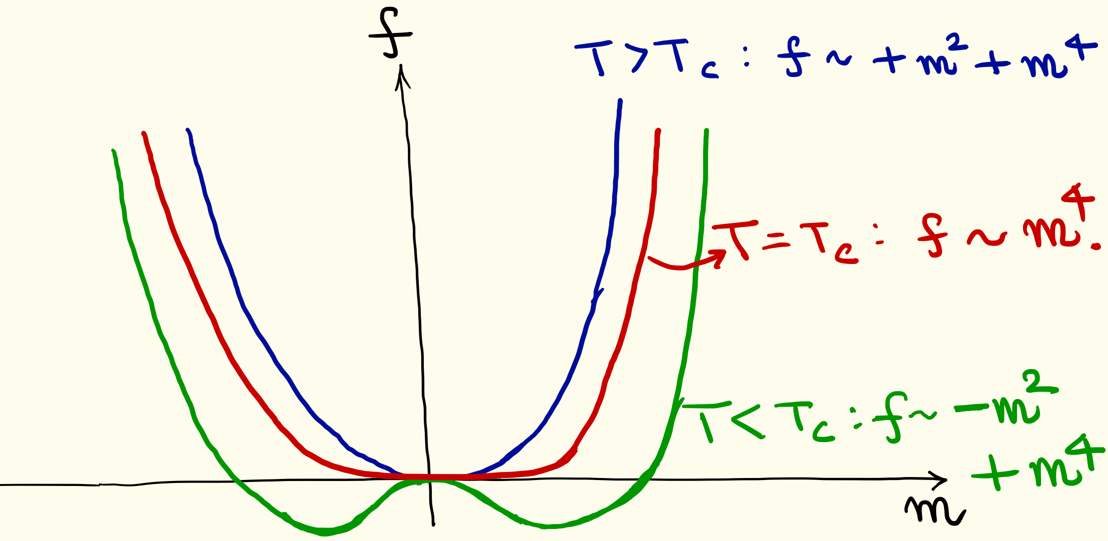
② In taylor expansion, the two lowest terms consistent with $m \to -m$ sym. are indeed m^2 and m^4, as seen above.

③ Most importantly, the coefficient of m^2 term <u>changes sign</u> at $T = T_c$.

The m^2 term is $\frac{Jm^2 z}{2}[1 - \beta J z]$

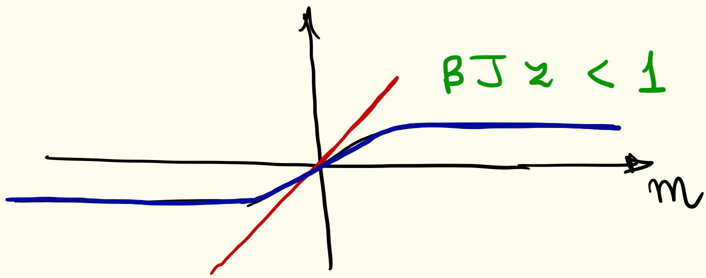
$$= \frac{Jm^2 z}{2}\left[1 - \frac{T_c}{T}\right].$$

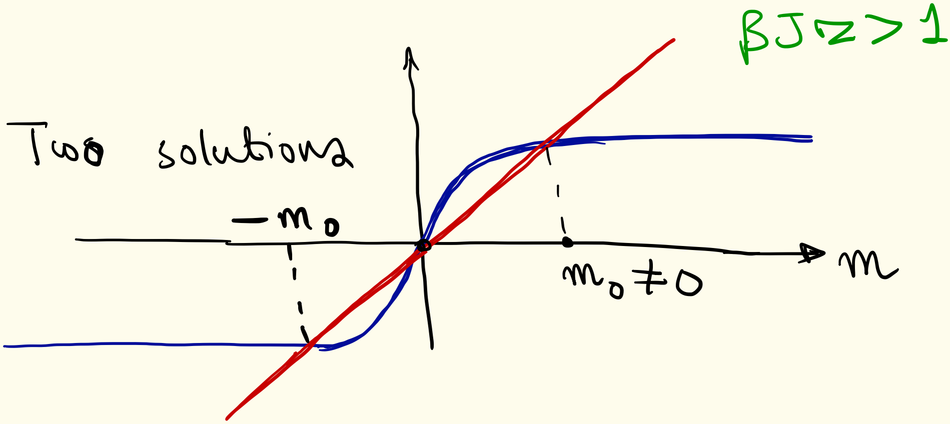
Let's plot $\frac{F}{N}$!



Because of the sign change of the m^2 term at $T = T_c$, f has global minima for $m \neq 0$ and a local maximum at $m=0$ for $T<T_c$. This is why there were two solutions to $\frac{\partial f}{\partial m} = 0$ eqn.

Recall the following figure:





The actual solution will be m_0 if $h = 0^+$ and $-m_0$ if $h = 0^-$

(h = external magnetic field).

Plotting f as a fn of m is thus very insightful. Now we can free ourselves from the burden of calculating partition fns and just write down the free energy based on symmetries:

When $h=0$, $m \to -m$ sym:

$$f = r m^2 + u m^4$$

$$r = \alpha (T - T_c)$$

So that for $T > T_c \Rightarrow r > 0$

$\Rightarrow$ $m=0$ only minima of m

$\Rightarrow$ no SSB.

Similarly for $T < T_c \Rightarrow r < 0$

$\Rightarrow$ $m \neq 0$ global minima (see fig. on last page)

$\Rightarrow$ <u>SSB</u>.

This quick and dirty approach goes by the name of Landau Theory.

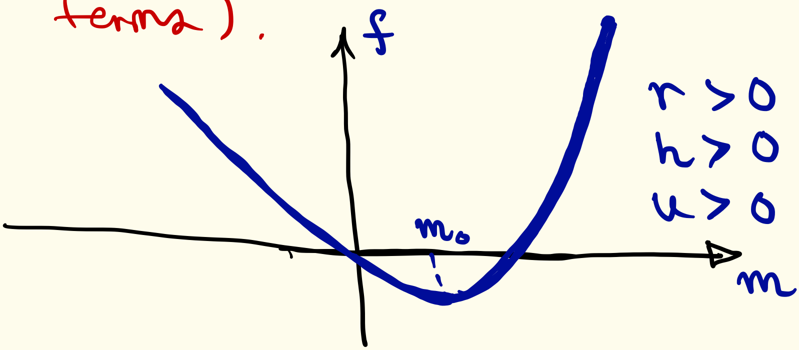
Let's redo this analysis with $h \neq 0$.

Now there is no $m \to -m$ sym. $\Rightarrow$ all terms allowed in f:

$$f = rm^2 - hm + um^4 \quad (h>0, u>0)$$

Now even when $r>0$, one finds a global minimum at $\boxed{m = \frac{h}{2r}}$.

(assuming $m \ll 1$, we keep only the lowest order terms).



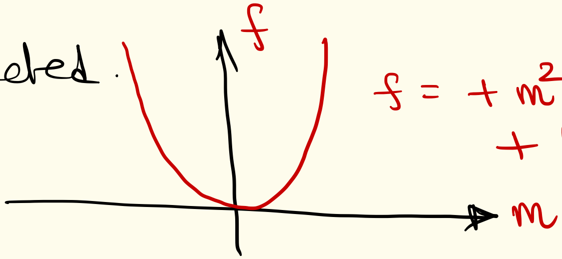
Thus when $h \neq 0$, m is always non-zero.

Let's also consider the limit

$h \to 0$.

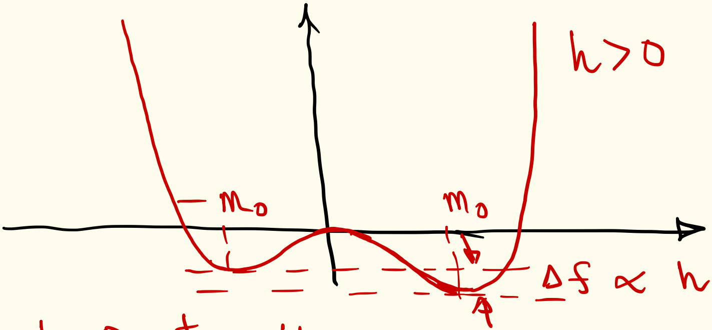
Nothing changes for $T > T_c$, $m \to 0$

as expected.



$f = +m^2 + m^4 + hm$ and $h \to 0$.

But when $T < T_c$



When $h \to 0^+$, the global soln. is $+m_0$. Similarly when $h \to 0^-$, it is $-m_0$. One might wonder if h is really going towards zero why is one soln preffered over the other?

The reason is that one is working in the limit.

$$\lim_{h \to 0^+} \lim_{N \to \infty} [\Delta F]$$

and not $$\lim_{N \to \infty} \lim_{h \to 0^+} [\Delta F]$$

where ΔF is the free-energy difference between the two minimas.

$$\Delta F \propto h N m_0$$

$$\Rightarrow \left. \begin{aligned} \lim_{h \to 0^+} \lim_{N \to \infty} [\Delta F] &= \infty \\ \lim_{N \to \infty} \lim_{h \to 0^+} [\Delta F] &= 0 \end{aligned} \right\}$$

This is the essence of Spontaneous Symmetry Breaking.