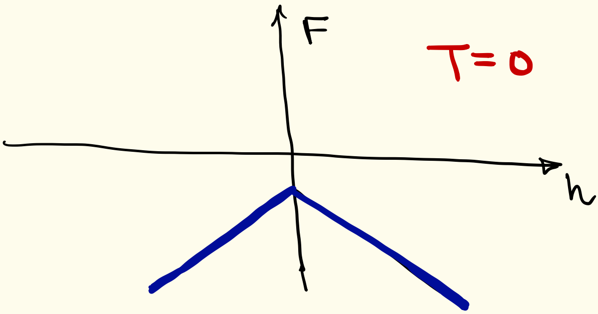
Summary so far:

1d Ising Model:

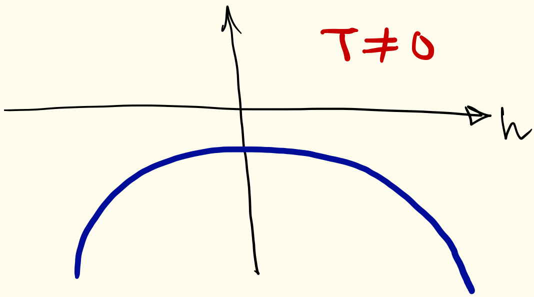
Non-zero magnetization at $T=0$ as $h \to 0$.

<u>Zero</u> ,, ,, $T \neq 0$ as $h \to 0$



$T=0$

$$M = \left.\frac{\partial f}{\partial h}\right|_{h=0^+} \neq 0$$



$T \neq 0$

$$M = \left.\frac{\partial f}{\partial h}\right|_{h=0^+} = 0$$

How can things look so different at $T=0$ vs $T \neq 0$? What if T is 10^{-10} K? Which of the above two plots will be the correct one. Let's analyze this in detail.

$$f = -T \log\left[\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right] - J$$

$$\Rightarrow m = -\frac{\partial f}{\partial h} = \beta T \frac{\left[\sinh(\beta h) + \frac{2\sinh(\beta h)\cosh(\beta h)}{2\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}\right]}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}$$

$$= \frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}$$

$$\lim_{\beta\to\infty} \lim_{h\to 0} m = \lim_{\beta\to\infty} \frac{0}{e^{-4\beta J}} = 0$$

$$\lim_{h\to 0} \lim_{\beta\to\infty} m = \lim_{h\to 0} \lim_{\beta\to\infty} \frac{\sinh(\beta h)}{|\sinh(\beta h)|} = \lim_{h\to 0} \operatorname{sign}(h)$$

$$= \operatorname{sign}(h)$$

i.e. $+1$ if $h \to 0^+$
-1 if $h \to 0^-$

Thus, limits don't commute.

Further, as already discussed, if $\beta \not\to \infty$, then in the limit $h \to 0$, m always go to zero.

Check: $\lim_{h \to 0} m(h) = \lim_{h \to 0} \frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}$

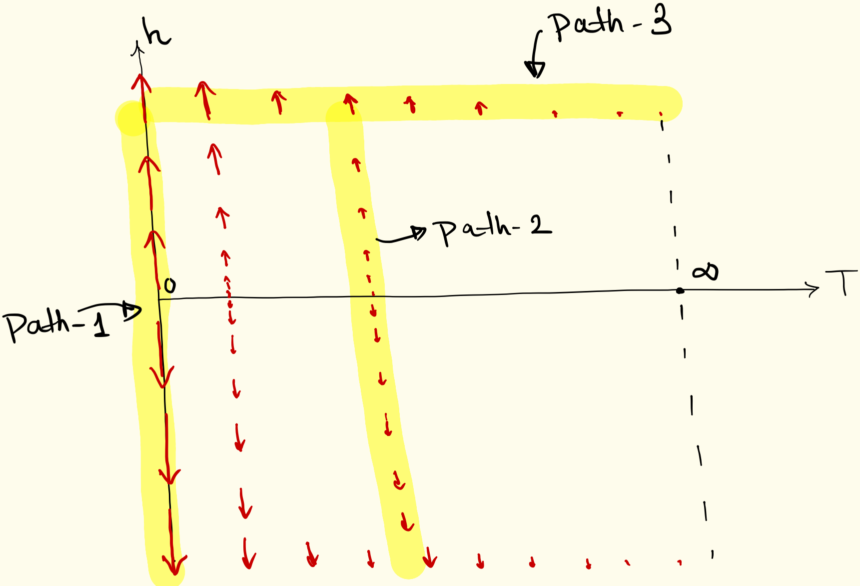
$$= 0$$

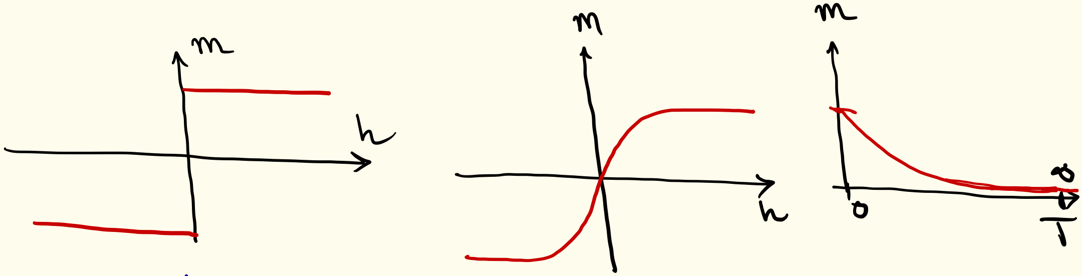
So if $T = 10^{-10}$ K and $h = 10^{-20}$ K, while $J = 1$K, then $m \simeq 0$ since $\beta h \ll 1$.

On the other hand, if $T = 10^{-20}$ K and $h = 10^{-10}$ K and $J = 1$K, then $m \simeq 1$ since $\beta h \gg 1$.

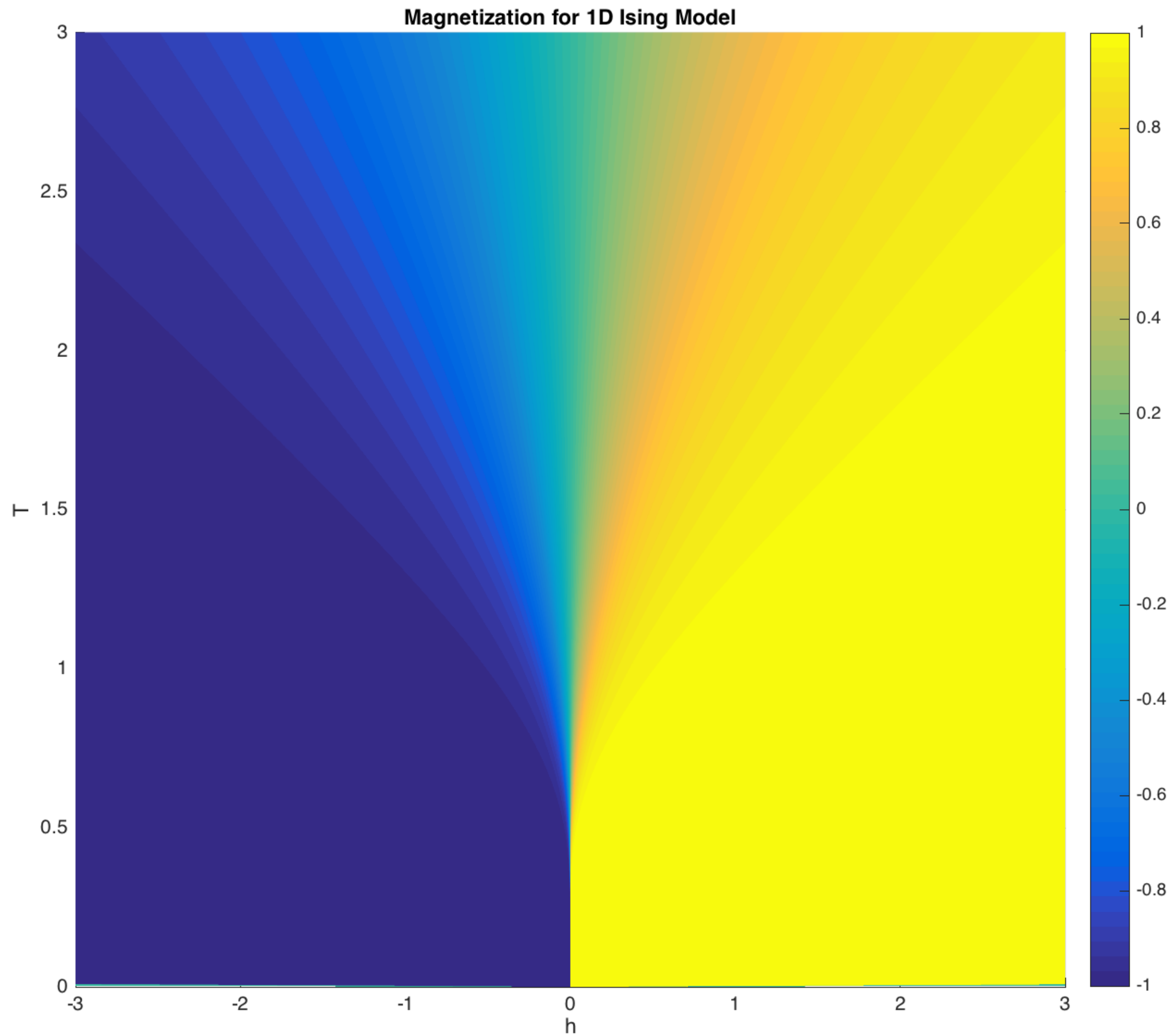
With this understanding, let's plot m in the h-T plane.

One really needs a 3d image to plot $m(T,h)$ but lets draw at a given (h,T) as an arrow whose direction denotes the sign of m and length the magnitude.





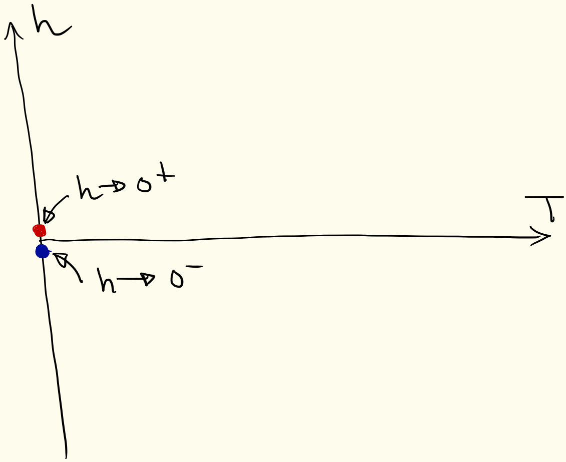
Path-1 Path-2 Path-3



Magnetization for 1D Ising Model
T
h
-3
-2
-1
0
1
2
3
0.5
1.5
2.5
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1

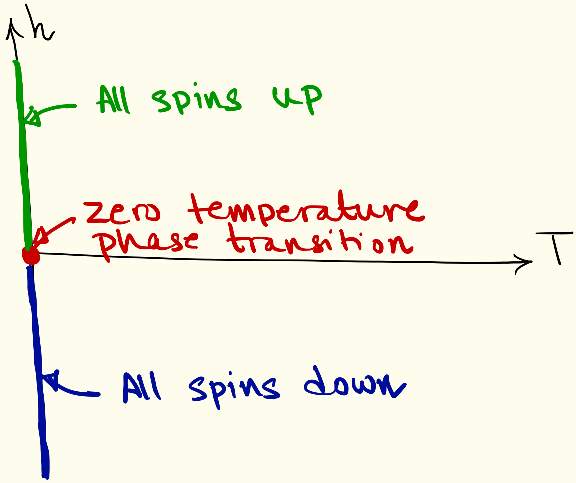
The singularity exists only at $(h, T) = 0$

Thus, the phase diagram looks like



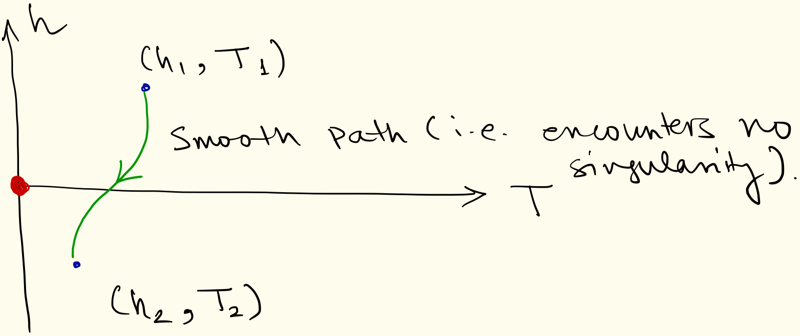
The two dots at the origin signify that the magnetization in the limit $h \to 0^+$ is not same as in the limit $h \to 0^-$.

Phase Diagram of 1d Ising Model



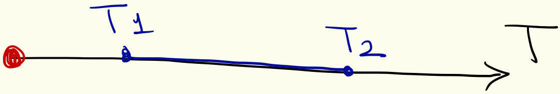
How many phases at $T > 0$?

Only one! Partition f^n singular only at the origin $\Rightarrow$ can go from any point on the (h, T) plane to any other point without encountering singularity.



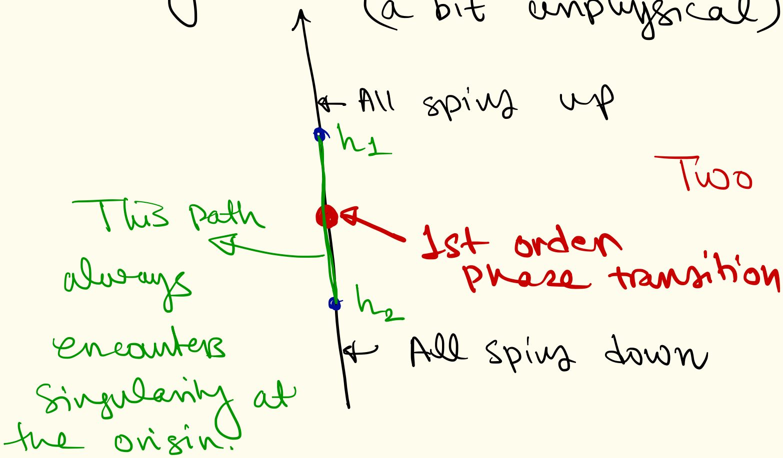
How many phases at $h = 0$?

(That is, while maintaining $S \to -S$ symmetry)



Only one.

How many phases at $T = 0$? (a bit unphysical)



Two phases!

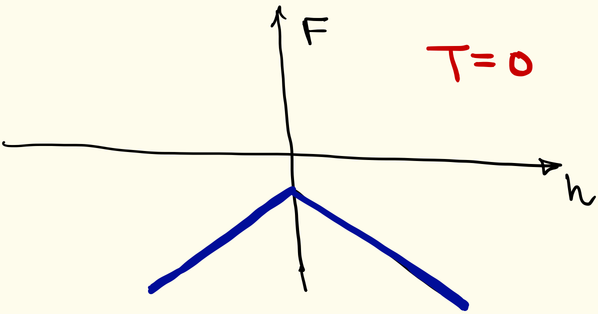
2. 2d Ising Model :

Non-zero magnetization at $T=0$ as $h \to 0$.

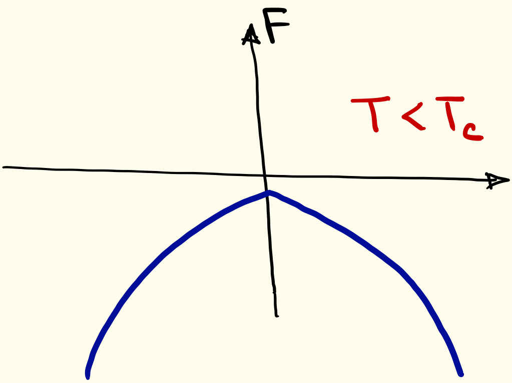
Non-zero ,, ,, $T < T_c$ as $h \to 0$

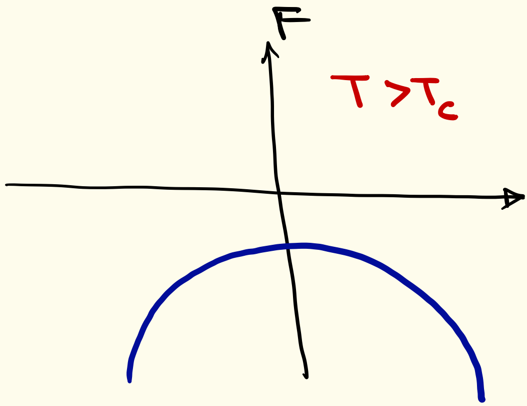
zero ,, ,, $T \geqslant T_c$ as $T \to 0$

$T_c \neq 0$



$$M = \left.\frac{\partial f}{\partial h}\right|_{h=0^+} \neq 0$$





How to see this? Mean-field Theory (see below)

Finally, when $h \neq 0$ (and not infinitesimal), magnetization is always non-zero.

How can singular behavior exist in the limit $h \to 0$ at finite T

Answer: Due to Thermodynamic limit $N \to \infty$. Recall the domain wall argument where

$$\Delta F_{\text{domain wall}} \sim \sqrt{N}\,[J - T\log(z-1)]$$

This is qualitative different for $J > T\log(z-1)$ vs $J < T\log(z-1)$ only when $N \to \infty$

More explicitly,

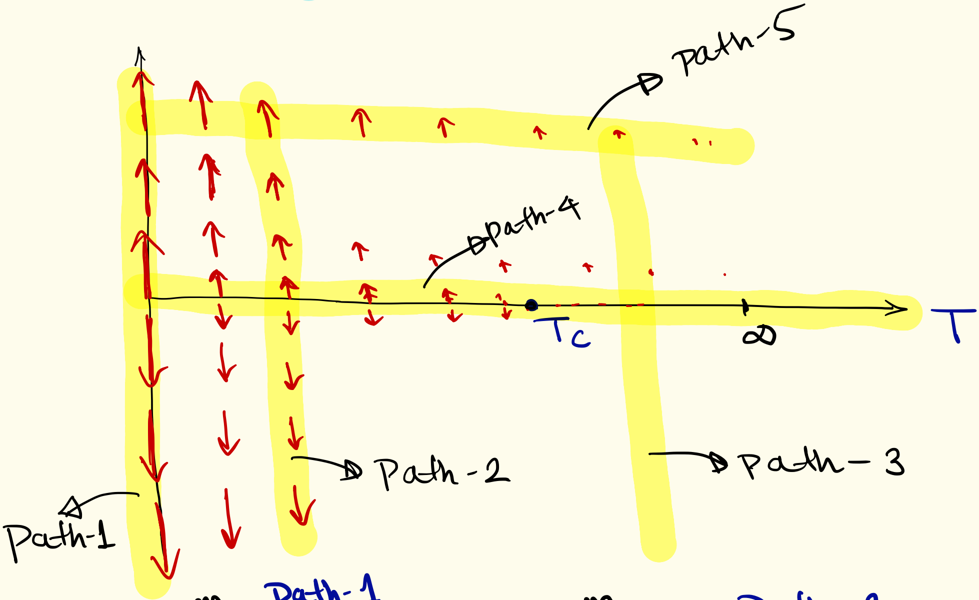
$$\boxed{\begin{gathered} \lim_{N\to\infty}\ \lim_{h\to 0}\ m(T,N,h) = 0 \ \text{ for all } T \\ \underline{\text{But}} \\ \lim_{h\to 0}\ \lim_{N\to\infty}\ m(T,N,h) \neq 0 \ \text{ for } T<T_c \end{gathered}}$$

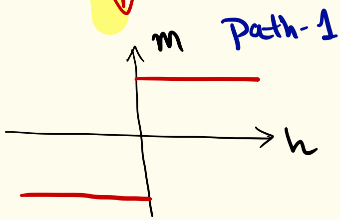
Thus, compared to 1d Ising, N plays the role of β to cause singular behavior.

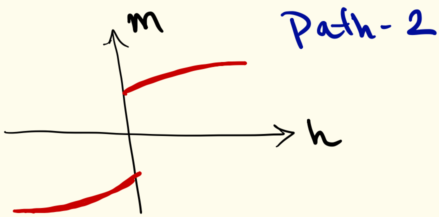
Let's again plot m in T, h plane

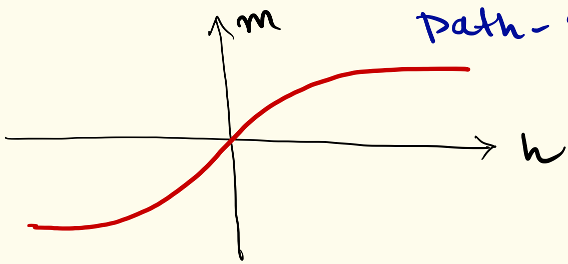
assuming that we first take the limit

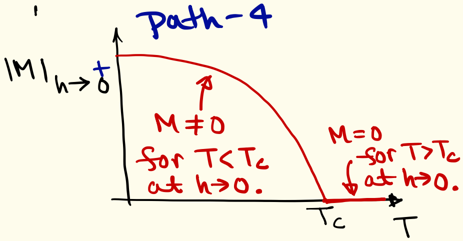
$N \to \infty$.

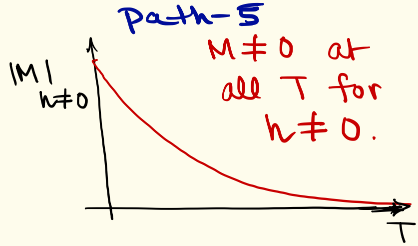




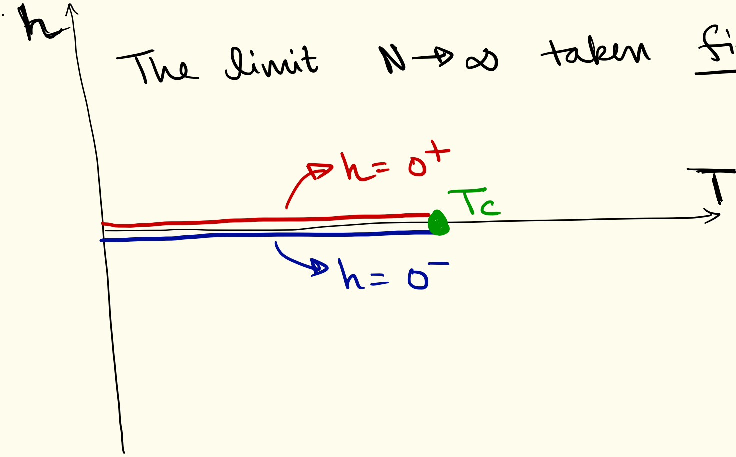






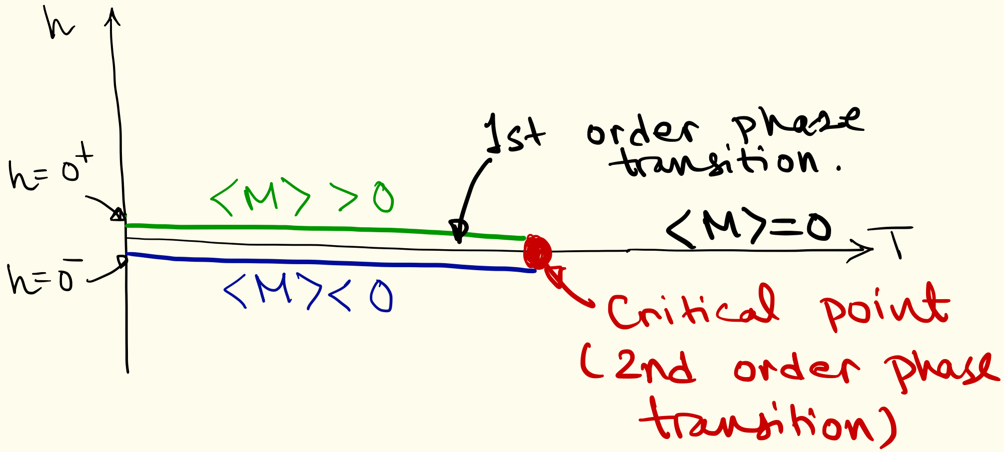


Phase Diagram:



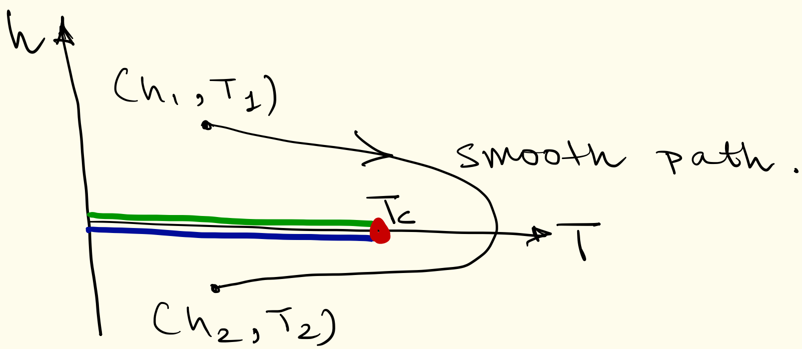
We will soon derive the qualitative features of the 2D Ising Model using Mean Field Theory

Phase Diagram of 2d Ising Model



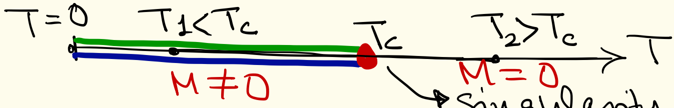
How many phases now?

At $T \neq 0$, if $h \neq 0$:



Only one phase when $h \neq 0$ at $T > 0$.

At $T \neq 0$, at $h = 0$ (i.e. while maintaining $S \to -S$ symmetry of H)



singularity at T_c.

Always encounter singularity when going from $T < T_c$ to $T > T_c$.

Two phases!

The low-T phase is called "Ordered" since $M \neq 0$. ($\equiv$ ferromagnetic phase)

The high-T phase is called "disordered" since $M = 0$.

($\equiv$ paramagnetic phase).