Exact Solution of 1d Ising Model

$$H = -J\sum_i s_i s_{i+1} - h\sum_i s_i$$

[Periodic boundary conditions]

$$Z = \sum_{\{s_i\}} e^{\beta J \sum_i s_i s_{i+1} + \beta h \sum_i s_i}$$

$$= \sum_{\{s_i\}} \prod_i e^{[\beta J s_i s_{i+1} + \beta h s_i]}$$

There is a smart way to do the above discrete sum:

$$Z = \sum_{\{s_i\}} \prod_i e^{\left(\beta J s_i s_{i+1} + \frac{\beta h}{2}(s_i + s_{i+1})\right)}$$

Think of $(i, i+1)$ as the row and column labels for a 2×2 matrix:

$$e^{\beta J s_i s_{i+1} + \frac{\beta h}{2}(s_i + s_{i+1})}$$

$$= \begin{array}{c|cc} & s_{i+1}=1 & s_{i+1}=-1 \\ \hline s_i=1 & e^{\beta(J+h)} & e^{-\beta J} \\ s_i=-1 & e^{-\beta J} & e^{\beta(J-h)} \end{array} \equiv M_{s_i, s_{i+1}}$$

$$\Rightarrow \quad Z = \sum_{\substack{s_1, s_2 \\ \dots s_N}} M_{s_1 s_2} M_{s_2 s_3} \text{---} M_{s_N s_1}$$

$$= \quad Tr\left[M^N\right]$$

$$= \quad \lambda_1^N + \lambda_2^N$$

where λ_1, λ_2 = eigenvalues of M.

Let's find λ_1, λ_2.

$$\det \begin{bmatrix} \lambda - e^{\beta J} e^{\beta h} & -e^{-\beta J} \\ -e^{-\beta J} & \lambda - e^{\beta J} e^{-\beta h} \end{bmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - \lambda e^{\beta J}(e^{\beta h} + e^{-\beta h}) + e^{2\beta J} - e^{-2\beta J} = 0$$

$$\Rightarrow \quad \lambda_{1,2} = \frac{e^{\beta J}(e^{\beta h} + e^{-\beta h}) \pm \sqrt{e^{2\beta J}[e^{2\beta h} + e^{-2\beta h} + 2] - 4[e^{2\beta J} - e^{-2\beta J}]}}{2}$$

$$= e^{\beta J}\left[\cosh(\beta h) \pm \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right]$$

$$\Rightarrow \lambda_1 = e^{\beta J}\left[\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right]$$

$$\lambda_2 = e^{\beta J}\left[\cosh(\beta h) - \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right]$$

$$Z = \lambda_1^N \left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N \right]$$

as $N \to \infty$ $\left(\frac{\lambda_2}{\lambda_1}\right)^N \to 0$ since $\frac{\lambda_2}{\lambda_1} < 1$

$$\Rightarrow \quad e^{-\beta F} = \lambda_1^N$$

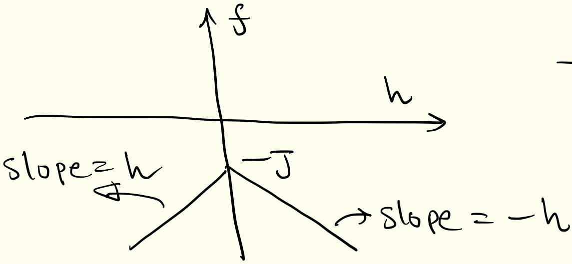
$$\Rightarrow \quad F = -TN \log(\lambda_1)$$

$$\Rightarrow \quad f = \frac{F}{N} = -T \log\left[\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right] - \frac{T \cdot J}{T}$$

In the limit $\beta J \gg 1 \gg \beta h$

$$f = -J - T \log\left[1 + T^{-1}\sqrt{h^2}\right]$$

$$\simeq -J - |h|$$



This is exactly as expected from the g.s. energy calculation.

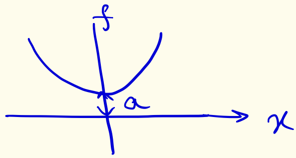
What about non-zero βh?

$\Rightarrow$ f is completely analytic as one may see from the exact expression:

$$f = -T \log\left[\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right] - J$$

<u>Why</u> is the above f^n analytic?

$\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}$ is analytic in βh as long as $e^{-4\beta J} \neq 0$ because the f^n

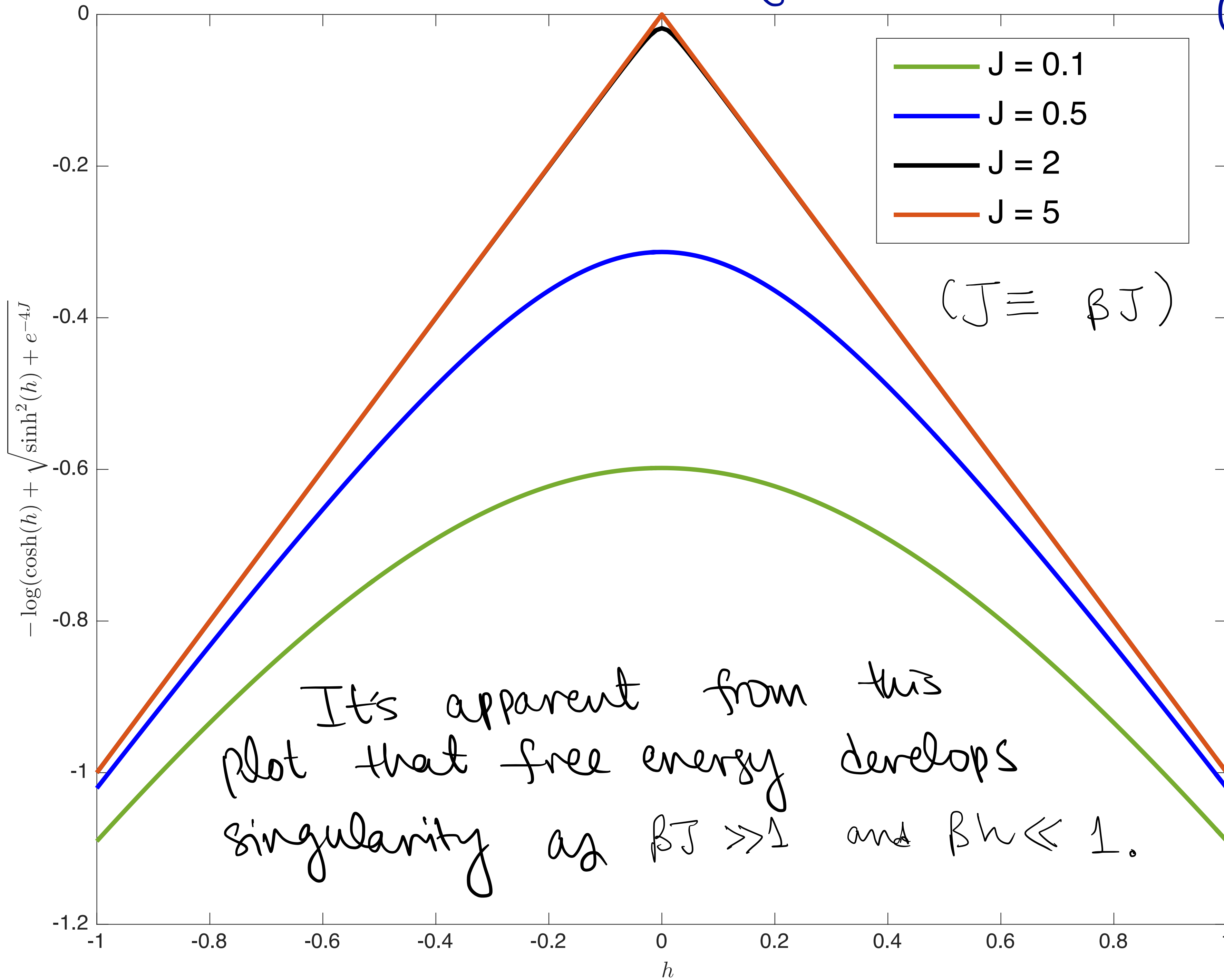
$\sqrt{x^2 + a^2}$ $\longrightarrow$ 

is analytic when $a \neq 0$.

$\log(g(x))$ is analytic as $g(x) > 0$ and analytic.

Both these conditions are satisfied above as long as βJ is <u>not</u> too large or βh not too small.

Plot of free energy dc ity vs h for 1d Ising



It's apparent from this plot that free energy develops singularity as $\beta J \gg 1$ and $\beta h \ll 1$.

Physical Properties of 1d Ising Model

Above we showed that the free energy density of the 1d Ising model is given by :

$$f = -T \log\left[\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}} \right] - J$$

Let's check some limits.

$T \to 0$: $f \to -J = \frac{E}{N}(T=0)$

$T \to \infty$: $f \to -T \log(2) = -T \frac{\log 2^N}{N}$

as expected.

$$\langle E \rangle = F + TS$$

$$= F - T \frac{\partial F}{\partial T}$$

$$= -T^2 \frac{\partial}{\partial T}\left[\frac{F}{T}\right] = -\frac{\partial}{\partial \beta}\left[\log Z\right]$$

Let's restrict ourselves to $h=0$.

Recall.

$$\lambda_1 = e^{\beta J}\left[\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right]$$

$$\lambda_2 = e^{\beta J}\left[\cosh(\beta h) - \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right]$$

$$\log[Z] = N \log \lambda_1$$

At $h=0$, $\lambda_{1,2} = e^{\beta J}\left[1 \pm e^{-2\beta J}\right]$

$$= e^{\beta J} \pm e^{-\beta J}$$

$$\Rightarrow \lambda_1 = 2\cosh(\beta J), \quad \lambda_2 = 2\sinh(\beta J)$$

$\Rightarrow$ At $h=0$,

$$E = -\frac{\partial}{\partial \beta}\left[\log Z(h=0)\right]$$

$$= -N \frac{\partial}{\partial \beta} \log\left(2\cosh(\beta J)\right)$$

$$= -NJ \tanh(\beta J)$$

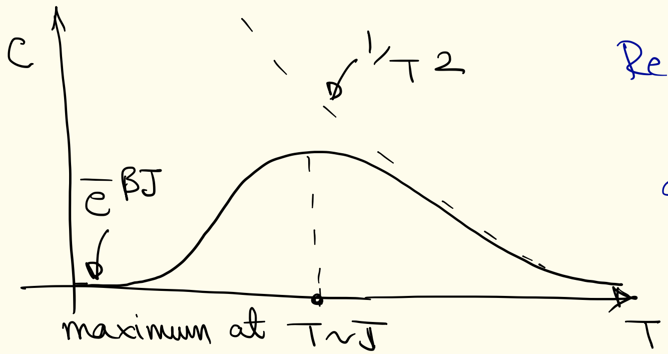
Specific heat $C = dE/dT$

$$= +\frac{NJ^2}{T^2}\operatorname{sech}^2(\beta J)$$

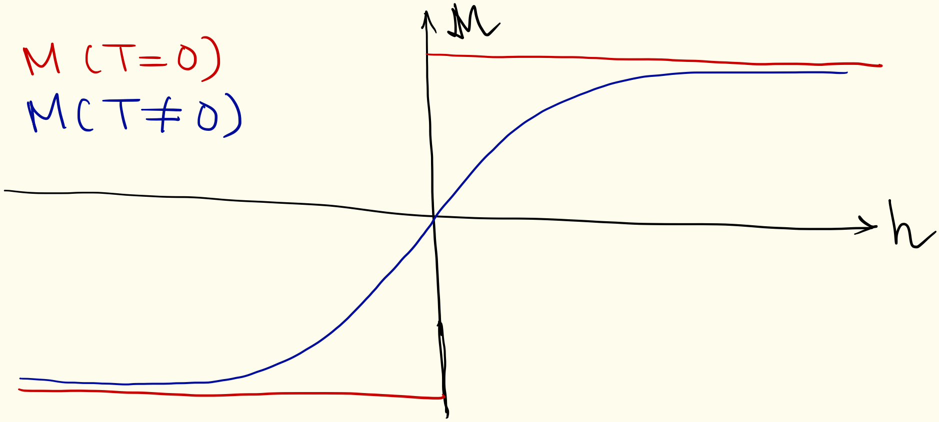
This is completely <u>analytical</u> f^n at $T \neq 0$.

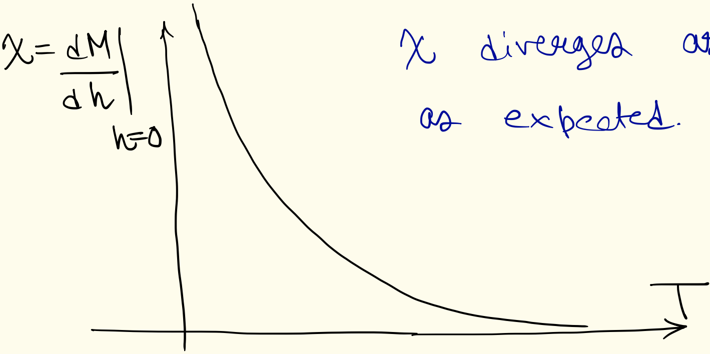
As $T \to 0$ $\quad C \sim \dfrac{e^{-J/T}}{T^2}$

As $T \to \infty$ $\quad C \sim \dfrac{1}{T^2}$



Recall: $e^{-\beta J}$ is <u>not</u> analytic at $\beta = \infty$.





χ diverges as $T \to 0$ (only)
as expected.