# Exact Calculation for the Specific Heat of the Fermi Gas

The total number of particles, N, is given by:

$$N = \sum_{\vec{p}\sigma} n_{\vec{p}\sigma}$$

$$= 2 \frac{V}{(2\pi)^3} \int \frac{4\pi k^2 \, dk}{e^{\beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)} + 1}$$

$$= 4\pi \left[\frac{2m}{h^2}\right]^{3/2} V \int_0^\infty \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1}$$

$$\Rightarrow \varepsilon_F^{3/2} = \int_0^\infty \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1}$$

Similarly, the total energy E is:

$$E = \sum_{\vec{p}\sigma} n_{\vec{p}\sigma} \varepsilon_{\vec{p}\sigma} = \sum_{\vec{p}\sigma} n_{\vec{p}\sigma} \frac{p^2}{2m}$$

$$= 4\pi \left[\frac{2m}{h^2}\right]^{3/2} V \int_0^\infty \frac{\varepsilon^{3/2}\, d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1}$$

Both of the above expressions are of the form

$$I = \int_0^\infty \frac{g(\varepsilon)\, d\varepsilon}{e^{(\varepsilon-\mu)\beta} + 1}$$

Since we are interested in the properties of the system only at temperatures $T \ll \varepsilon_F$, one can do a Taylor's expansion for the integral $I$. Let's see how it goes:

(see appendix A.13 of Garrod's book for similar discussion, although our discussion will be self-contained anyway).

First we note that at $T = 0$ (i.e. $\beta = \infty$)

$I$ becomes $I_0 = \int_0^\infty g(\varepsilon)\, \theta\left(\frac{\mu-\varepsilon}{T}\right) d\varepsilon$

$$\left( = \int_0^\mu g(\varepsilon)\, d\varepsilon \right)$$

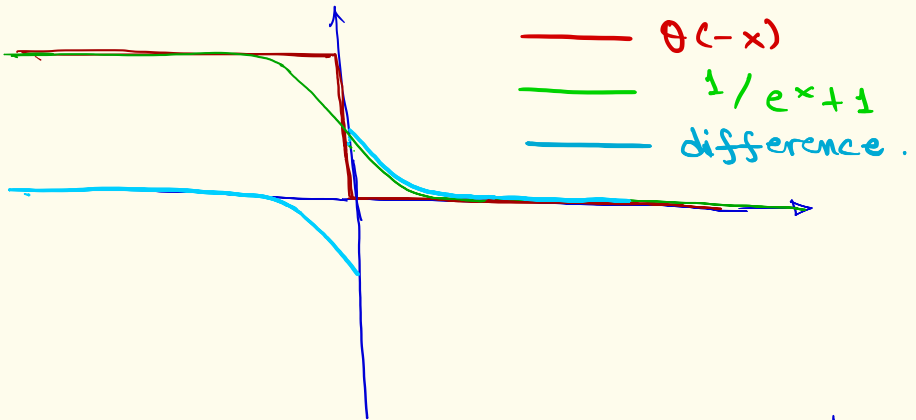
Let's consider the difference $\Delta I = I - I_0$.

$$\Delta I = \int_0^\infty \left[ \frac{1}{e^{(\varepsilon-\mu)/T} + 1} - \theta\left(\frac{\mu-\varepsilon}{T}\right) \right] g(\varepsilon)\, d\varepsilon$$

This is a bit messy. To make it look nicer, let's consider change of variables to the dimensionless parameter $x = (\varepsilon - \mu)/T$.

$$\Delta I = T \int_{-\mu/T}^{\infty} \left[ \frac{1}{e^x+1} - \theta(-x) \right] g(\mu + Tx) \, dx$$

The function $\frac{1}{e^x+1} - \theta(-x)$ looks like




Thus the difference approaches zero exponentially fast when $|x| \gg 1$. Since the lower limit of integration $= -\mu/T \ll 0$ at low temperatures (the regime of our interest), one can safely extend the limits of integration to $-\infty$. The error incurred under this approximation would be of the order of $e^{-\mu/T}$ which is really small e.g. when $\mu/T \approx 100$, then error $\approx e^{-100}$.

Having made this approximation, one can now taylor expand:

$$\Delta I = T \int_{-\infty}^{\infty} \left[ \frac{1}{e^x+1} - \theta(-x) \right] g(\mu + Tx)$$

$$= \sum_{n=0}^{\infty} T^{n+1} \left. \frac{d^n g(\varepsilon)}{d\varepsilon^n} \right|_{\varepsilon=\mu} \int_{-\infty}^{\infty} x^n \left[ \frac{1}{e^x+1} - \theta(-x) \right]$$

The function $f = \frac{1}{e^x+1} - \theta(-x)$ is an odd

fn of $x$ as may be obvious from the figure on the previous page. But lets check this explicitly:

If $x > 0$ $\quad f(x) = \frac{1}{e^x+1}$

$$f(-x) = \frac{1}{e^{-x}+1} - 1 = \frac{-1}{e^x+1} = -f(x)$$

Similarly, one can check $x < 0$.

$\Rightarrow \quad x^n \left[ \frac{1}{e^x+1} - \theta(-x) \right]$ is an

even $f^n$ of $x$ when $n$ is odd and

an odd $f^n$ of $x$ when $n$ is even.

Since the limits of integration in $\Delta I$ are from $-\infty$ to $+\infty$, this implies that only odd values of $n$ contribute.

$$\Delta I = \sum_{n \text{ odd}} T^{n+1} \left. \frac{d^n g(\varepsilon)}{d\varepsilon^n} \right|_{\varepsilon=\mu} \times 2 \times \underbrace{\int_0^\infty \frac{x^n \, dx}{e^x+1}}_{\shortparallel}$$

$$= 2 \sum_{n \text{ odd}} T^{n+1} \frac{d^n g(\varepsilon)}{d\varepsilon^n} \Big| \underbrace{\left[1 - 2^{-n}\right] n! \, \zeta(n+1)}$$

How to do the integral $\displaystyle\int_0^\infty \frac{x^n}{e^x+1} dx$ ?

$$\int_0^\infty \frac{x^n}{e^x+1}\,dx = \int_0^\infty \frac{x^n\, e^{-x}}{1+e^{-x}}\,dx$$

$$= \sum_{m=0}^{\infty} \int_0^\infty x^n\, e^{-x}\, e^{-mx}\, (-)^m\, dx$$

$$= \sum_{m=1}^{\infty} \frac{1 \quad n!}{m^{n+1}}\, (-)^{m+1}$$

This looks a bit like Riemann-zeta $f^n$ but has the oscillating terms due to $(-)^m$. Fret not!

Note that $\displaystyle\sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m^{n+1}} - \zeta(n+1)$

$$= \sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m^{n+1}} - \sum_{m=1}^{\infty} \frac{1}{m^{n+1}}$$

$$= -2\sum_{m=2,4,6,\ldots} \frac{1}{m^{n+1}} = \frac{-2}{2^{n+1}} \sum_{m=1}^{\infty} \frac{1}{m^{n+1}}$$

$$\Rightarrow \sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m^{n+1}} = \zeta(n+1)\left[1 - 2^{-n}\right]$$

$$\Rightarrow \int_0^\infty \frac{x^n}{e^x+1}\,dx = n!\, \zeta(n+1)\left[1-2^{-n}\right]$$

Using the known values of Riemann-zeta fn for integers n, one thus find,

$$I = \int_0^{\mu} g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6} g'(\mu) T^2$$

$$+ \frac{7\pi^4}{360} g''(\mu) T^4 + \cdots$$

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Let's apply the above formula to the problem at hand, namely specific heat of fermions at low temperatures.

$$N = 4\pi \left[\frac{2m}{h^2}\right]^{3/2} V \int_0^{\infty} \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

$$= 4\pi \left[\frac{2m}{h^2}\right]^{3/2} V \left[ \int_0^{\mu} \sqrt{\varepsilon}\, d\varepsilon + \frac{\pi^2}{6} \frac{1}{2\sqrt{\mu}} T^2 + \cdots \right]$$

$$\Rightarrow \underbrace{\frac{N}{4\pi V} \left[\frac{h^2}{2m}\right]^{3/2} \times \frac{3}{2}}_{\varepsilon_F^{3/2}} = \mu^{3/2} + \frac{\pi^2 T^2}{8\sqrt{\mu}}$$

Can replace $\mu$ by $\varepsilon_F$ at this order.

$$\Rightarrow \quad \varepsilon_F^{3/2} = \mu^{3/2} + \frac{\pi^2 T^2}{8\sqrt{\varepsilon_F}}$$

$$\Rightarrow \quad \mu = \left[ \varepsilon_F^{3/2} - \frac{\pi^2 T^2}{8\sqrt{\varepsilon_F}} \right]^{2/3}$$

$$= \varepsilon_F \left[ 1 - \frac{\pi^2 T^2}{8 \varepsilon_F^2} \right]^{2/3}$$

$$\Rightarrow \quad \mu = \varepsilon_F - \frac{\pi^2 T^2}{12 \varepsilon_F}$$

Recall that in the ramp approximation,

$$\mu = \varepsilon_F - \frac{1}{4} \frac{D'(\varepsilon_F)}{D(\varepsilon_F)} (\delta\varepsilon)^2$$

$$\approx \varepsilon_F - \frac{9}{8} \frac{T^2}{\varepsilon_F} \quad \text{if } \delta\varepsilon = 3T$$

a bit arbitrary.

Let's next calculate the total energy exactly:

$$E = 4\pi \left[ \frac{2m}{h^2} \right]^{3/2} V \int_0^\infty \frac{\varepsilon^{3/2}\, d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1}$$

$$\Rightarrow \frac{1}{4\pi} \left( \frac{E}{V} \right) \left[ \frac{h^2}{2m} \right]^{3/2} = \int_0^\mu \varepsilon^{3/2}\, d\varepsilon + \frac{\pi^2}{6} \times \frac{3}{2} \sqrt{\mu}\, T^2$$

$$= \frac{2}{5} \mu^{5/2} + \frac{\pi^2}{4} \mu^{1/2} T^2$$

$$= \frac{2}{5} \left[ \varepsilon_F - \frac{\pi^2 T^2}{12 \varepsilon_F} \right]^{5/2} + \frac{\pi^2}{4} \left[ \varepsilon_F - \frac{\pi^2 T^2}{12 \varepsilon_F} \right]^{1/2} T^2$$

$$= \frac{2}{5} \varepsilon_F^{5/2} \left[ 1 - \frac{\pi^2 T^2}{12 \varepsilon_F^2} \times \frac{5}{2} \right]$$

$$+ \frac{\pi^2}{4} \varepsilon_F^{1/2} T^2 \left[ 1 - \frac{\pi^2 T^2}{12 \varepsilon_F} \times \frac{1}{2} \right]$$

$$= \frac{2}{5} \varepsilon_F^{5/2} + \varepsilon_F^{1/2} T^2 \left[ \frac{\pi^2}{4} - \frac{\pi^2}{12} \right]$$

Now note that

$$\frac{1}{4\pi} \left( \frac{E}{V} \right) \left[ \frac{h^2}{2m} \right]^{3/2} = \frac{E}{N} \times \frac{2}{3} \varepsilon_F^{3/2}$$

$$\Rightarrow \frac{E}{N} = \frac{2}{5} \varepsilon_F^{5/2} \times \frac{3}{2} \varepsilon_F^{-3/2}$$

$$+ \frac{T^2 \pi^2}{6} \varepsilon_F^{1/2} \times \frac{3}{2} \varepsilon_F^{-3/2}$$

$$\Rightarrow \frac{E}{N} = \frac{3}{5} \varepsilon_F + \frac{\pi^2 T^2}{4 \varepsilon_F}$$

$$\Rightarrow C_{V,N} = \left. \frac{dE}{dT} \right|_{V,N} = \frac{\pi^2}{2} \frac{TN}{\varepsilon_F}$$

Reformulation in terms of density of states:

$$D(\varepsilon_F) \sim \frac{V\, m^{3/2}\, \varepsilon_F^{1/2}}{h^3}$$

$$\sim \frac{N}{\varepsilon_F}$$

since $$\frac{N}{V} \sim \frac{m^{3/2}\, \varepsilon_F^{3/2}}{h^3}$$

$$\Rightarrow \quad c_{V,N} \sim T\, D(\varepsilon_F)$$

Putting exact pre-factors:

$$c_{V,N} = \frac{\pi^2}{3}\, T\, D(\varepsilon_F)$$