Identical Bosons without Boson number conservation

There exist identical bosons whose total number is not constrained by any laws of nature. The most prominent example is photons — for example an atom in an excited state can undergo a transition to a state of lower energy and emit a photon in the process (to conserve energy). Since the photon number is not conserved, the concept of canonical ensemble is meaningless — the only conserved quantity is the total energy (whose conjugate 'chemical potential' is β^{-1}) and there is no chemical potential conjugate to the total photon number, which is allowed to be anything. One has to sum over all possible photon occupation numbers.

A photon with momentum $\vec{p}$ carries an energy $c|\vec{p}|$. In addition, photon carries a polarization which can take two values (say, ± 1).

Assuming that the photons are non-interacting (this is a good approximation), the single particle energies are: $\varepsilon_{\vec{p}\sigma} = c|\vec{p}|$ where $\sigma = \pm 1$ denotes polarization.

Recall that the grand partition f^n is given by

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta \hat{H}}$$
$$= \sum_{\{n_{p\sigma}\}} e^{-\beta \sum_{p\sigma} n_{p\sigma} \varepsilon_{p\sigma}}$$

where $n_{p\sigma}$ is the occupation of the single particle level with momentum $\vec{p}$ and polarization σ. We already did this exercise earlier when we discussed identical bosons in general. The result is:

$$\mathcal{Z} = \prod_{\vec{p}} \left[\frac{1}{1 - e^{-\beta c p}} \right]^2$$ where the

square stems due to two allowed polarizations, both of which have same single particle energy.

The grand potential Ω is thus given by

$$\Omega = -\frac{1}{\beta} \log \mathcal{Z}$$
$$= +\frac{1}{\beta} \sum_{\vec{p}} 2 \log \left[1 - e^{-\beta c p} \right]$$

In the thermodynamic limit, the single particle levels are closely spaced and we can convert the sum into an integral.

Assuming that the photons are localized in a box of size V, $\sum_k \longrightarrow \frac{V}{(2\pi)^3}\int d^3k$ with $p = \hbar k$.

$$\Rightarrow \quad \Omega = \frac{2}{\beta}\frac{V}{(2\pi)^3}\int d^3k \log\left[1-e^{-\beta c\hbar k}\right]$$

$$= \frac{2}{\beta}\frac{V}{(2\pi)^3} 4\pi \int_0^\infty dk\, k^2 \log\left[1-e^{-\beta c\hbar k}\right]$$

Define $\quad x = \beta c\hbar k$

$$\Rightarrow \quad \Omega = \frac{2}{\beta}\frac{V}{(2\pi)^3}\frac{4\pi}{[\beta c\hbar]^3}\int_0^\infty dx\, x^2 \log\left[1-e^{-x}\right]$$

Doing integration by parts,

$$\Omega = -\frac{2}{\beta}\frac{V}{(2\pi)^3}\frac{4\pi}{[\beta c\hbar]^3}\frac{1}{3}\int_0^\infty dx \frac{x^3 e^{-x}}{1-e^{-x}}$$

$$= -\frac{2}{\beta}\frac{V}{(2\pi)^3}\frac{4\pi}{[\beta c\hbar]^3}\frac{1}{3}\int_0^\infty dx \frac{x^3}{e^x-1}$$

The integral $\displaystyle\int_0^\infty dx \frac{x^a}{e^x-1} = \zeta(a+1)\,\Gamma(a+1)$

where $\Gamma(a+1)$ is the Gamma f^n ($= a!$ for integer 'a')
and $\zeta(a+1) = \sum_{n=1}^\infty \frac{1}{n^{a+1}}$ is the famous Riemann-zeta f^n.

For the case at hand, $\int_0^\infty \frac{x^3}{e^x - 1} = \zeta(4)\,\Gamma(4)$

$$= \frac{\pi^4}{90} \times 3!$$

$$= \frac{\pi^4}{15}$$

$$\Rightarrow \Omega = -\frac{2}{\beta} \frac{V}{(2\pi)^3} \frac{4\pi}{[\beta c \hbar]^3} \frac{1}{3} \times \frac{\pi^4}{15}$$

$$\Rightarrow \boxed{\Omega = -\frac{8}{45} \pi^5 \frac{T^4 V}{h^3 c^3}}$$

Recall $\Omega = -PV$

$$\Rightarrow \boxed{P = \frac{8\pi^5}{45} \frac{T^4}{h^3 c^3}}$$

Since $d\Omega = -S\,dT - P\,dV$

$$\Rightarrow S = -\frac{\partial \Omega}{\partial T}$$

Further, $E = TS - PV = TS + \Omega$

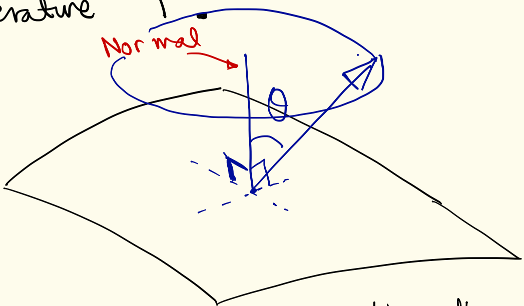
$$= -T \frac{\partial \Omega}{\partial T} + \Omega$$

$$\Rightarrow \frac{E}{V} = \frac{8\pi^5}{15} \frac{T^4}{h^3 c^3} = 3P$$

Black Body Radiation

A black body emits the same amount as it absorbs. This is equivalent to the statement that it absorbs all of the radiation impinged on it.

Let's calculate the energy emitted by a blackbody at temperature T.



The power emitted in a direction θ to the surface normal is:

$$\frac{dP}{dA} = c \frac{E}{V} \cos\theta \frac{d\Omega}{4\pi}$$

where $d\Omega$ is the solid angle and E is the energy calculated above.

$$\Rightarrow \frac{dP}{dA} = \frac{cE}{4\pi V} \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \sin\theta \cos\theta$$

$$= \frac{cE}{4V} = \frac{2\pi^5}{15} \frac{T^4}{h^3 c^2} = \sigma T^4$$

where $\sigma = 5.67 \times 10^{-8}\ W\, m^{-2}\, K^{-4}$

Alternative way to calculate Photon energy:

$$E = \underset{\substack{\uparrow \\ \text{Two polarizations}}}{2} \times \sum_{\vec{k}} \underset{\substack{\uparrow \\ \text{Energy of photon at momentum } \vec{k}}}{\hbar c k} \times \underset{\substack{\uparrow \\ \text{Bose occupation factor for photon at } \vec{k}}}{\frac{1}{e^{\beta \hbar c k} - 1}}$$

$$= \frac{2 V}{(2\pi)^3} \hbar c \int \frac{k \cdot 4\pi k^2 \, dk}{e^{\beta c \hbar k} - 1}$$

Clearly scales as T^4.

$$= \frac{2}{(2\pi)^3} \frac{\hbar c \times 4\pi V}{[\beta c \hbar]^4} \underbrace{\int \frac{x^3 \, dx}{e^x - 1}}_{= \frac{\pi^4}{15}}$$

$$= \frac{8\pi^5}{15} \frac{T^4 V}{h^3 c^3}.$$

Fun Question: Estimate the surface temperature of Earth knowing the surface temperature of sun and geometrical details (radius of earth & sun, distance between earth and sun). See Dan Arovas lecture notes.

$$T_{earth} = T_{sun} \times \left[\frac{\text{Radius of Sun}}{2 \times \text{Radius of Earth's orbit}}\right]^{1/2}$$

High Temperature expansion for free bosons and fermions.

The grand free energy is given by

$$\beta \mathcal{G} = -\beta P V$$

$$= -\eta \sum_{k} \log \left[\frac{1}{1 - \eta\, e^{-\beta\left[\frac{k^2}{2m} - \mu\right]}} \right]$$

where $\eta = +1$ for bosons and -1 for fermions.

$$\Rightarrow \quad \beta P = -\frac{\eta}{8\pi^3} \int d^3k \, \log\left[1 - \eta\, e^{-\frac{\beta k^2}{2m}} z\right]$$

where $z = e^{\beta\mu}$

The value of z is determined by requiring that the total number of particles is N.

$$\Rightarrow \quad N = \sum_{\vec{k}} \frac{1}{e^{\frac{\beta k^2}{2m}} z^{-1} - \eta}$$

$$\Rightarrow \quad \rho = \frac{N}{V} = \frac{1}{8\pi^3} \int d^3k \, \frac{1}{e^{\beta k^2/2m} z^{-1} - \eta}$$

In the high-temperature limit, from classical statistical mechanics, we expect that μ will be large and negative $\Rightarrow z = e^{\beta\mu}$ will be very small. This motivates one to do the above integrals by tayor expanding in powers of z. But let's first 'non-dimensionalize' the above integral.

$$\rho = \frac{1}{2\pi^2}\int_0^\infty \frac{k^2\,dk}{e^{\beta k^2/2m} z^{-1} - \eta}$$

$$= \frac{1}{\lambda^3}\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{dx\,\sqrt{x}}{e^{x} z^{-1} - \eta} \qquad \left[\text{by substituting } \frac{\beta k^2}{2m} = x\right]$$

$$\Rightarrow \rho\lambda^3\sqrt{\pi}/2$$

$$= \int_0^\infty \frac{dx\,\sqrt{x}\;e^{-x} z}{[1 - \eta e^{-x} z]} \qquad \left(\begin{array}{c}\lambda = \text{de-Broglie}\\ \text{thermal}\\ \text{length}\end{array}\right)$$

$$= \int_0^\infty dx\,\sqrt{x}\;e^{-x} z \sum_{m=0}^{\infty}[\eta e^{-x} z]^m$$

$$= \int_0^\infty dx\,\sqrt{x}\sum_{m=1}^{\infty}(e^{-x} z)^m\,\eta^{m-1}$$

$$= \sum_{m=1}^{\infty}\int dx\,\sqrt{x}\;e^{-mx}\,z^m\,\eta^{m-1}$$

$$= \sum_{m=1}^{\infty} \eta^{m-1} \frac{z^m}{m^{3/2}} \quad \Gamma(3/2)$$

$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$$

$$= \frac{\sqrt{\pi}}{2} \left[z + \frac{\eta z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \cdots \right]$$

$$\Rightarrow \rho \lambda^3 = z + \frac{\eta z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \cdots$$

This equation can be solved recursively.

For example, to the lowest order, $z^{(1)} = \rho \lambda^3$.

At next order $z^{(2)} = \rho \lambda^3 - \frac{\eta}{2^{3/2}} \left[z^{(1)} \right]^2$

$$= \rho \lambda^3 - \frac{\eta}{2^{3/2}} \left[\rho \lambda^3 \right]^2$$

and so on...

Let's keep terms to order $(\rho \lambda^3)^2$ so that

$$z \simeq \rho \lambda^3 - \frac{\eta}{2^{3/2}} \left[\rho \lambda^3 \right]^2 .$$

Let's use this value of $z \; (= e^{\beta \mu})$ to calculate the pressure at high temperatures.

The equation for pressure is,

$$\beta P = -\frac{\eta}{8\pi^3} \int d^3k \log\left[1 - \eta z e^{-\frac{\beta k^2}{2m}}\right]$$

$$= \frac{\eta}{8\pi^3} 4\pi \int_0^\infty \frac{dk\, k^3}{3} \frac{\eta z e^{-\beta k^2/2m}}{1-\eta z e^{-\beta k^2/2m}} \frac{\beta k}{m}$$

(integration by parts)

$$= \frac{1}{\lambda^3} \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{3/2}}{e^x z^{-1} - \eta} \quad \left[\frac{\beta k^2}{2m} = x\right]$$

Again Taylor expanding in z:

$$\frac{3\sqrt{\pi}}{4} \beta P \lambda^3 = \int_0^\infty dx\, x^{3/2} e^{-x} z \sum_{m=0}^{\infty} (\eta e^{-x} z)^m$$

$$= \sum_{m=1}^{\infty} \eta^{m-1} \frac{z^m}{m^{5/2}} \Gamma(5/2)$$

Since $\Gamma(5/2) = \frac{3\sqrt{\pi}}{4}$

$$\Rightarrow \beta P \lambda^3 = z + \frac{\eta z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} + \cdots$$

Substituting $z \simeq \rho\lambda^3 - \frac{\eta}{2^{3/2}}[\rho\lambda^3]^2$

$$\Rightarrow \quad \beta P \lambda^3 = \rho\lambda^3 - \frac{\eta}{2^{3/2}}[\rho\lambda^3]^2 + \frac{\eta}{2^{5/2}}(\rho\lambda^3)^2 + \cdots$$

Since $\left(\frac{1}{2^{3/2}} - \frac{1}{2^{5/2}}\right) = \frac{1}{\sqrt{2}}\left[\frac{1}{2} - \frac{1}{4}\right]$

$$= \frac{1}{2^{5/2}}$$

$$\Rightarrow \beta P \lambda^3 = \rho\lambda^3 - \frac{\eta}{2^{5/2}}(\rho\lambda^3)^2 + \cdots$$

$$\Rightarrow \quad P = \rho T\left[1 - \frac{\eta\rho\lambda^3}{2^{5/2}} + \cdots\right]$$

Compare this with an ideal classical gas where. $P = \rho T$ $\Rightarrow$ for bosons ($\eta = +1$), the pressure is <u>less</u> compared to an ideal gas in this high-temperature regime, while for fermions ($\eta = -1$), the pressure exceeds that of a classical gas. This makes intuitive sense.