

Zippping many-body quantum states

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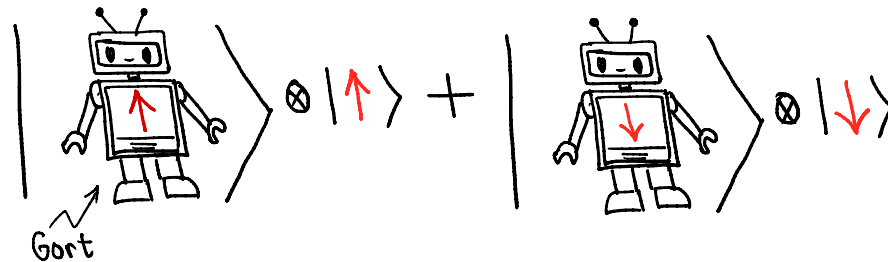
arXiv:2502.18898

In praise of entropy

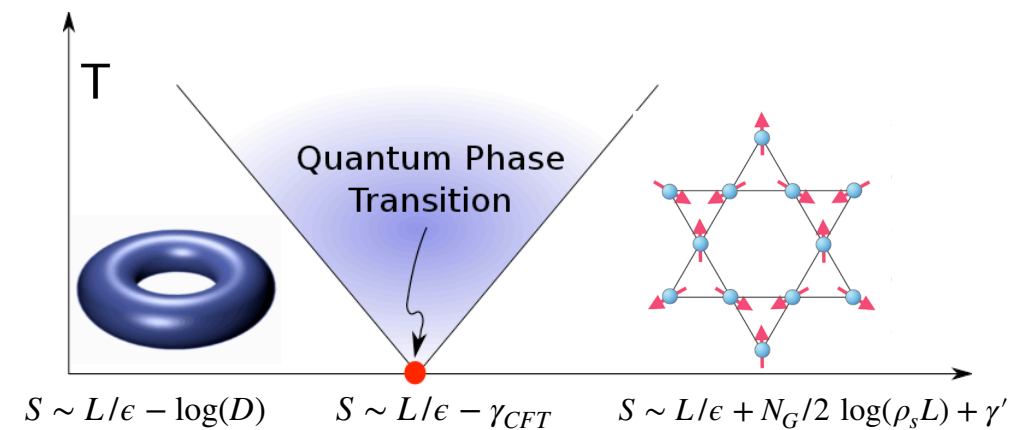
Information-theoretic objects of the form $S = -\text{trace}(\rho \log(\rho))$
have been extremely useful in a variety of contexts...



Fundamentals of
thermodynamics and
statistical mechanics



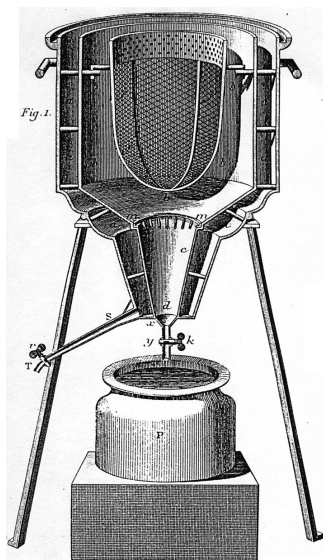
“Interpretation of classical
mechanics”



Universal aspects of many-body states

Measuring entropy?

$S = -\text{trace}(\rho \log(\rho))$ is the expectation value of an unwieldy operator $\log(\rho)$.



Ice calorimeter [Lavoisier, Laplace, circa 1780]

For a Gibbs state, $dE = S_{\text{thermal}} dT$

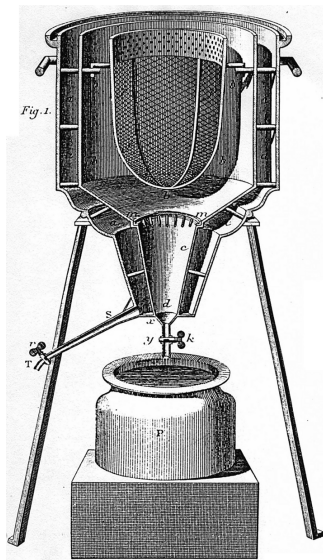
$\Rightarrow S_{\text{thermal}}(T)$ can be obtained from $E = \langle H(T) \rangle$

Ergodicity implies microcanonical $S(E) = S_{\text{thermal}}(T(E))$

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Entropy of a non-thermal subsystem? A calorimeter won't do...



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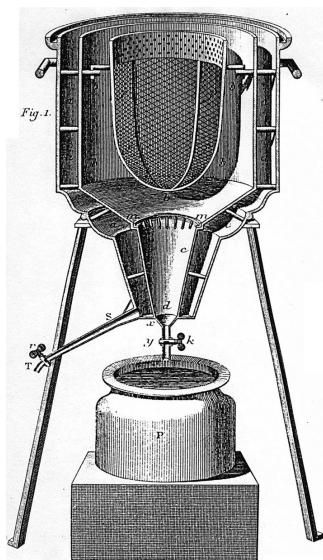
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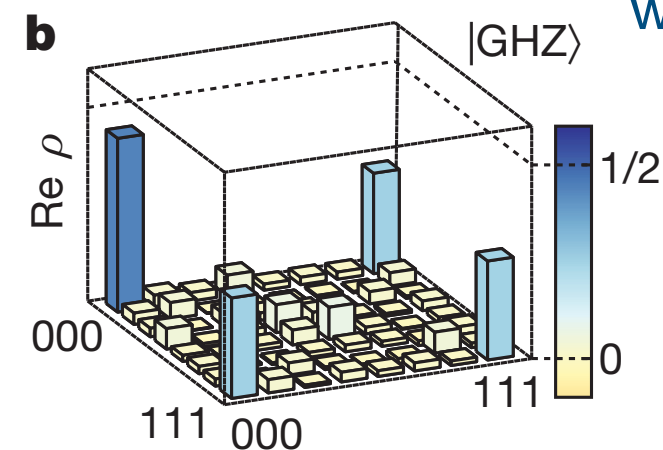
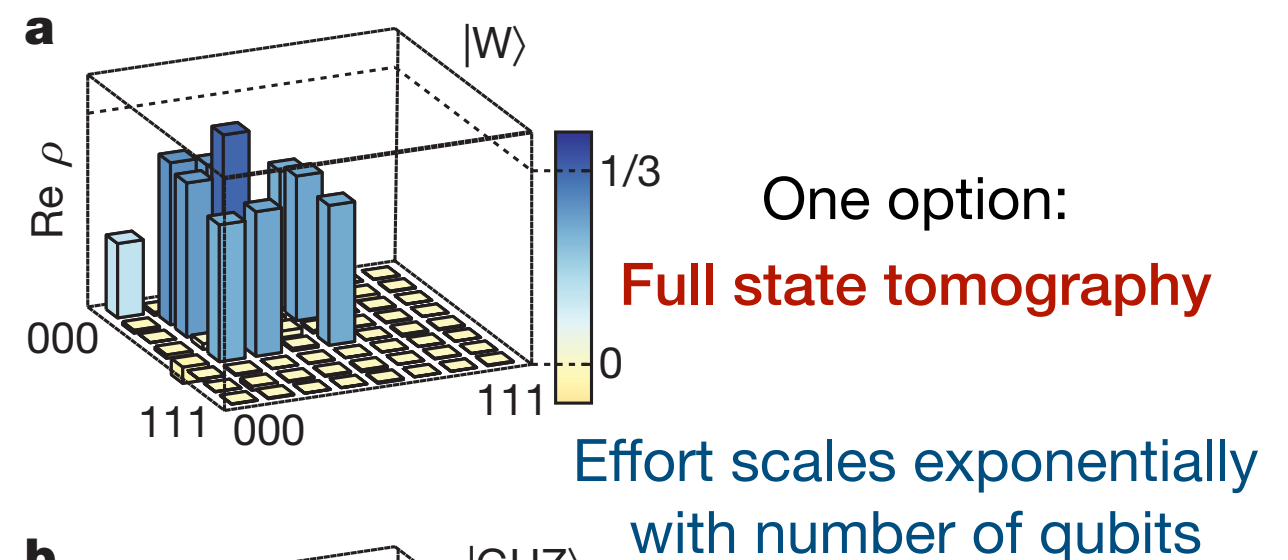


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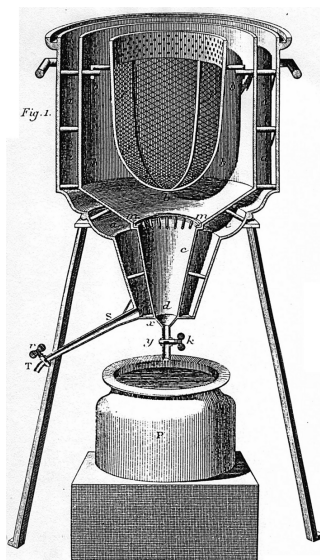


[Neeley,...,Martinis 2010]

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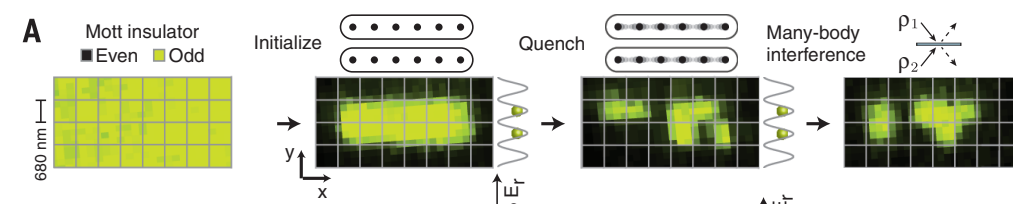
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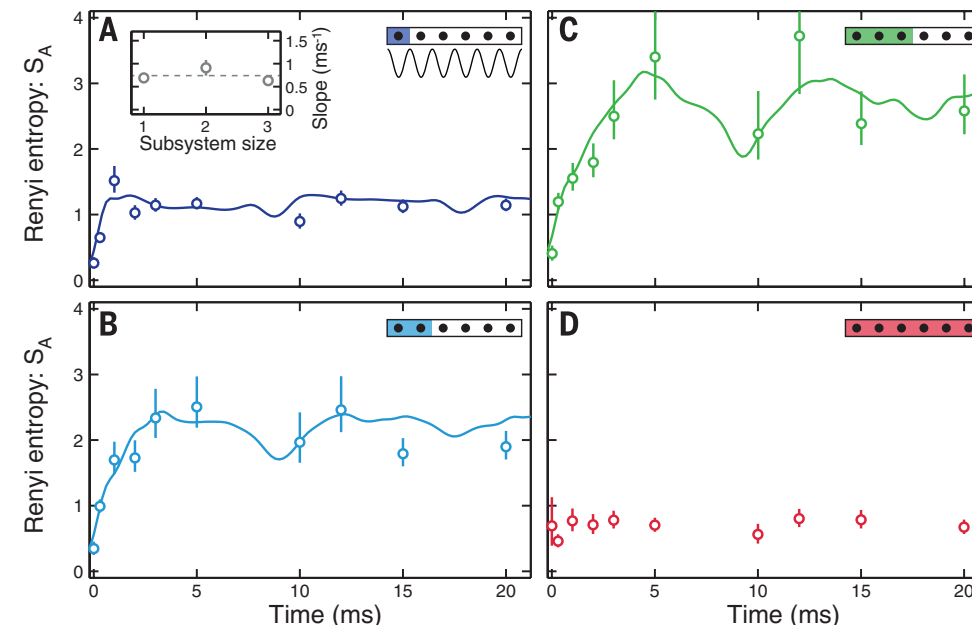
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Ergodicity implies microcanonical $S(E) = S_{\text{thermal}}(T(E))$

Measure Renyi entropy $S_2 = -\log(\text{tr}(\rho^2))$ instead...



Many-body twin interferometry



[Islam,..., Greiner 2015; Kaufman,..., Greiner 2016]

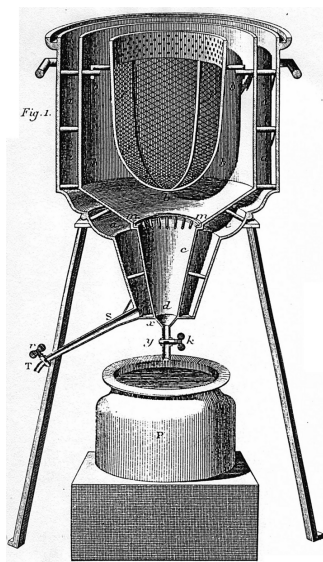
Scaling still unfavorable

Measuring entropy?

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Measure Renyi entropy $S_2 = -\log(\text{tr}(\rho^2))$ instead...

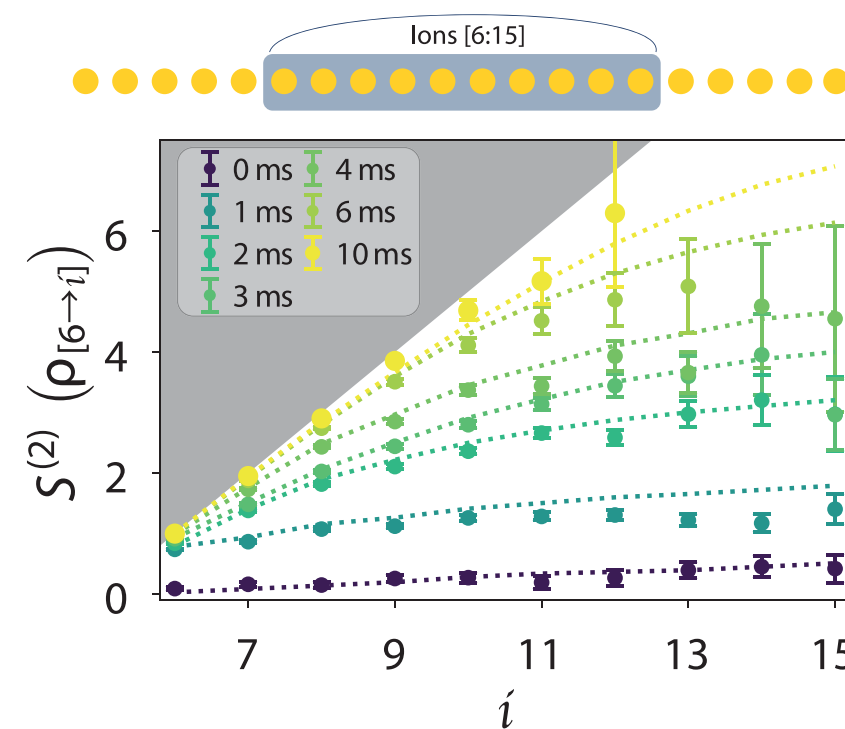


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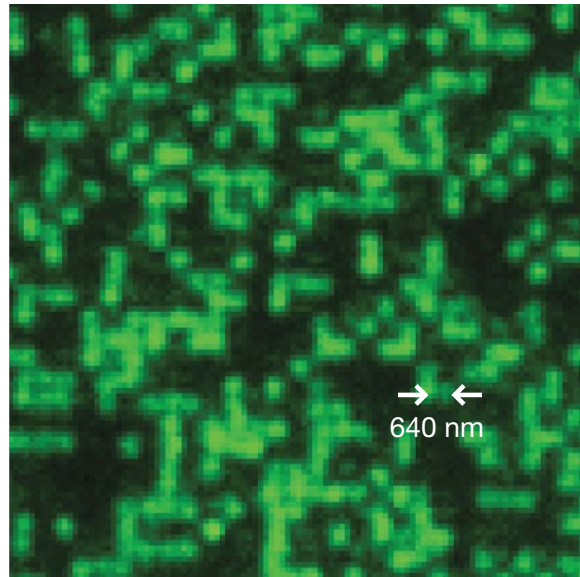


Randomized
unitary+ measurement
method

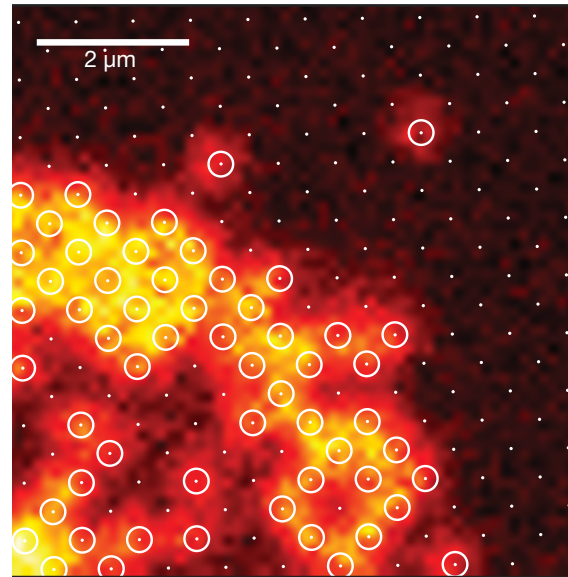
[Brydges,...,
Roos, 2020;
Aaronson 2018;
Huang, Kueng,
Preskill, 2020]

Scaling still
unfavorable

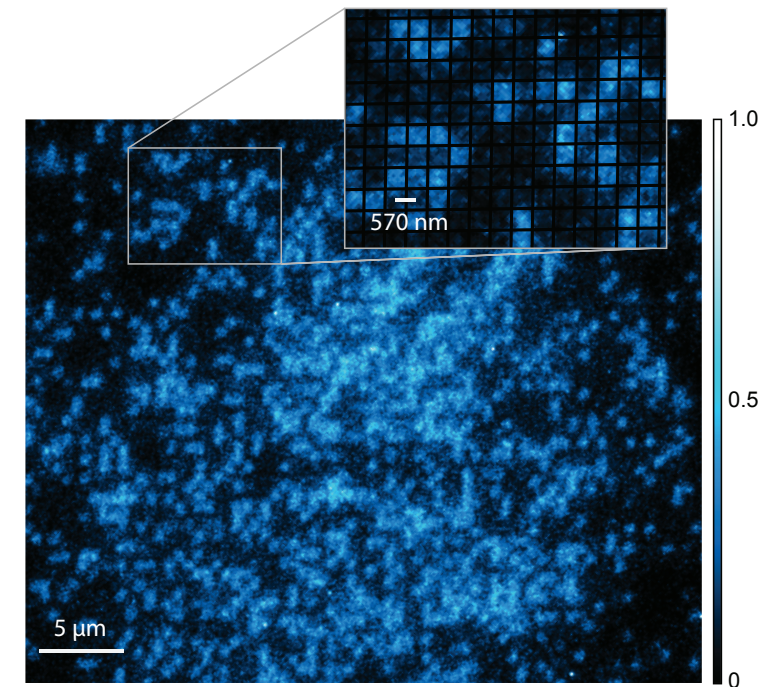
Measuring entropy?



Site-resolved image
of ^{87}Rb atoms
[Bakr,..., Greiner, 2010]



Site-resolved image
of ^{87}Rb atoms
[Sherson,..., Kuhr, 2010]



Site-resolved image
of ^6Li atoms
[Parsons,..., Greiner 2015]

$$\text{Probability of an image} = \langle \text{Image} | \rho | \text{Image} \rangle$$

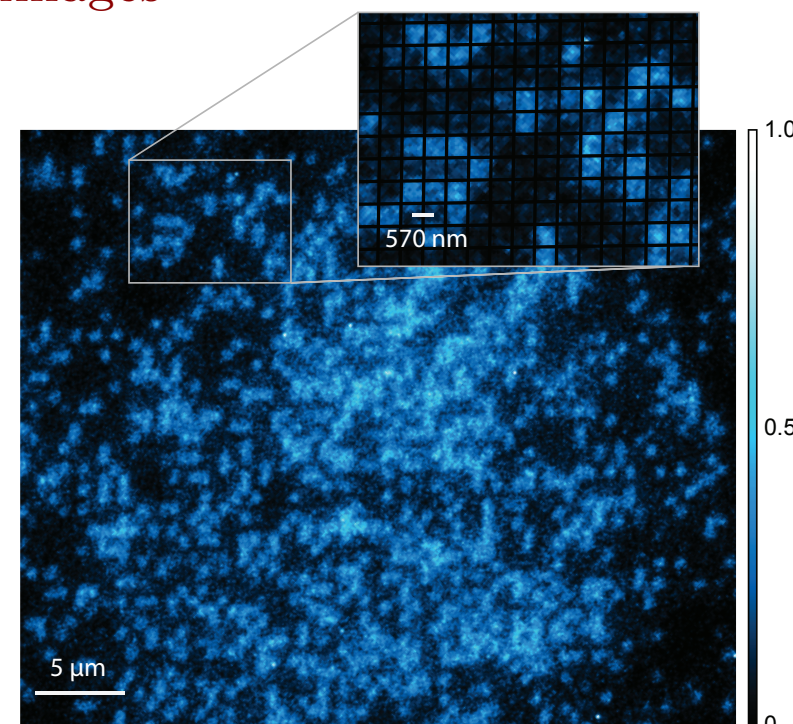
These images are precious. Generating them on a classical computer generically a hard task.

Can these be used to measure *any* entropic quantity efficiently at all?

Instead of targeting $S = -\text{trace}(\rho \log(\rho))$, we will ask a simpler question:

Can we estimate “diagonal entropy” S_d , defined as

$$S_d = - \sum_{\text{images}} \text{probability}(\text{image}) \cdot \log(\text{probability}(\text{image}))$$



Probability of an image

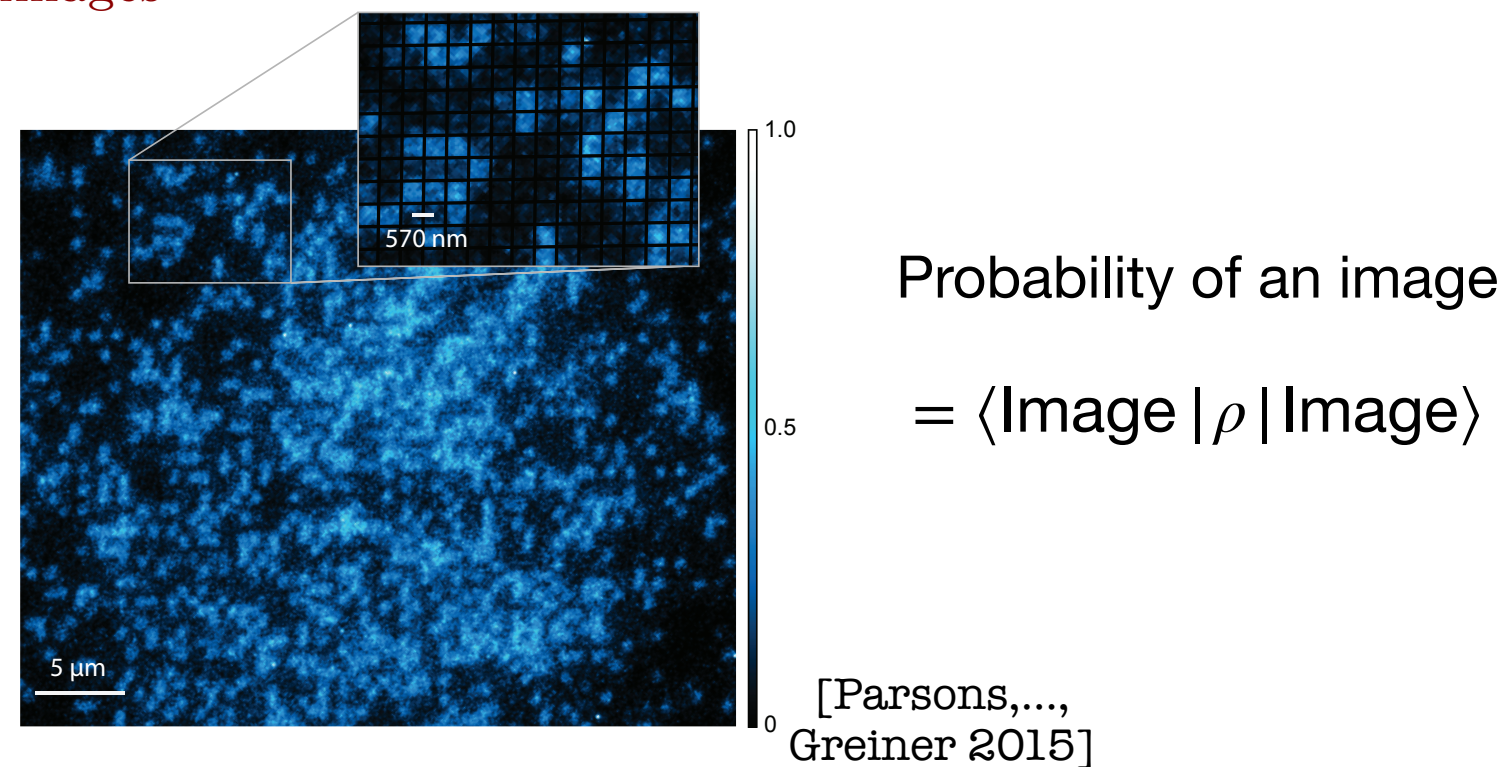
$$= \langle \text{Image} | \rho | \text{Image} \rangle$$

[Parsons,...,
Greiner 2015]

Instead of targeting $S = -\text{trace}(\rho \log(\rho))$, we will ask a simpler question:

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- Unlike S , diagonal entropy S_d is non-zero even for pure states.
- S_d generically scales as volume law, even for ground states.
- $S_d \geq S$.

Basic idea



$$\text{Diagonal entropy} = N \cdot \text{Compression ratio} = N \cdot \frac{1 \text{ MB}}{2 \text{ MB}} = N/2$$

Entropy and Data Compression

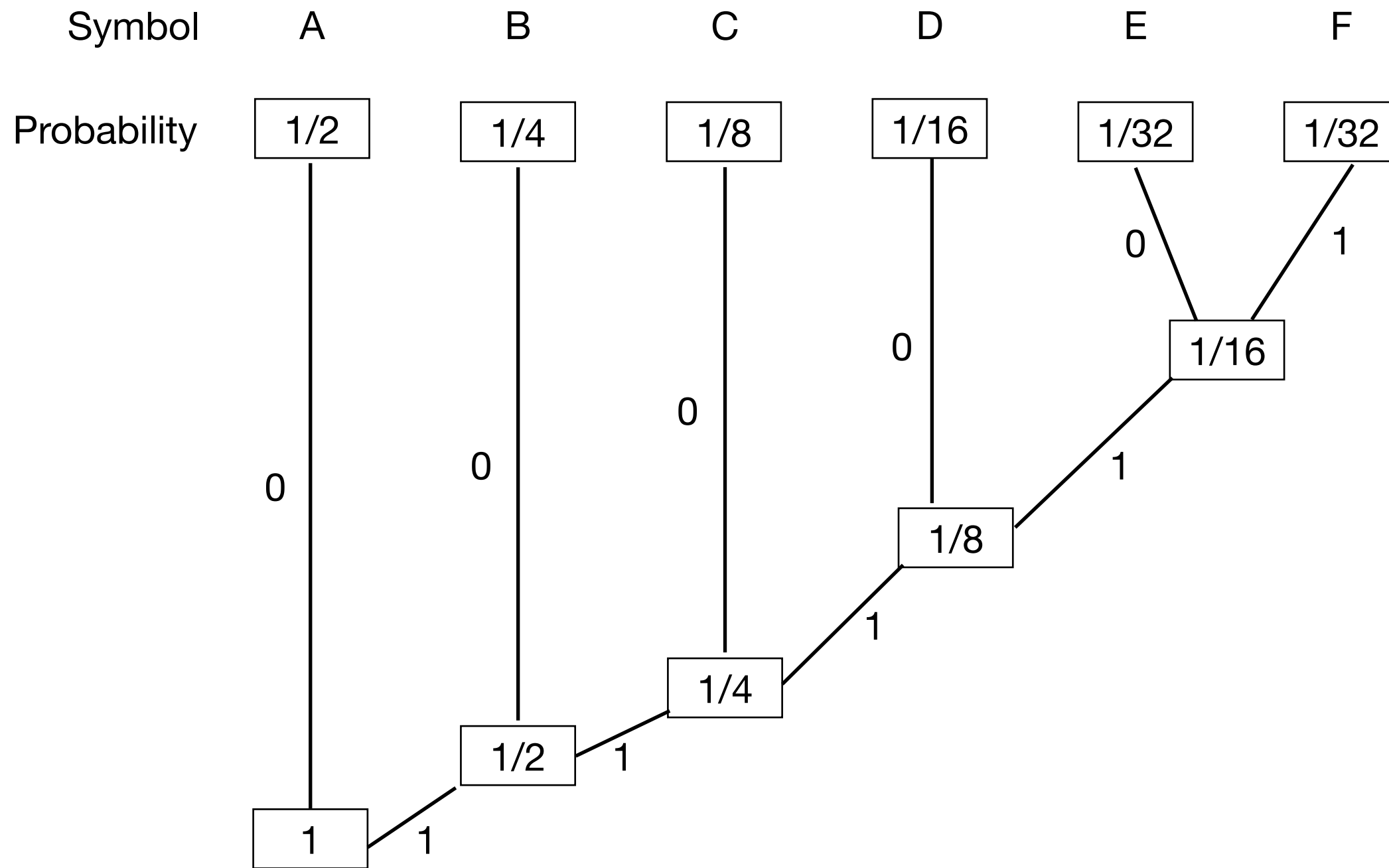
Most efficient way to compress a string? e.g., consider

AAABABAAABBBAAAABAAACAAADAAABEAAFFAABABAAABBBAAAABAAACAAADAAAB
EAAFFAABABAAABBBAAAABAAACAAADAAABEAAAF

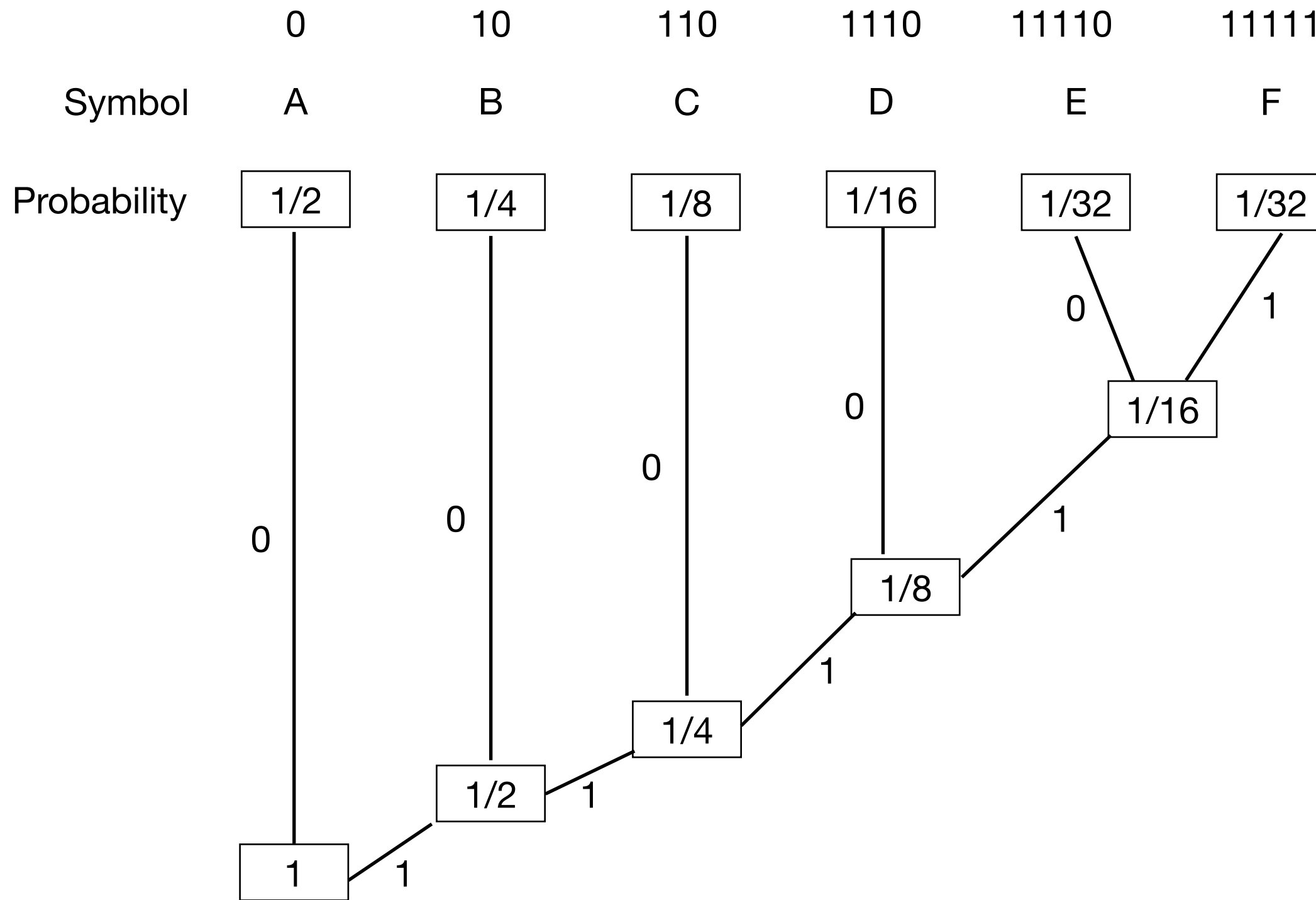
where the probabilities of these symbols are known a priori:

Symbol	A	B	C	D	E	F
Probability	1/2	1/4	1/8	1/16	1/32	1/32

Entropy and Data Compression



Entropy and Data Compression



Entropy and Data Compression

	0	10	110	1110	11110	11111
Symbol	A	B	C	D	E	F
Probability	1/2	1/4	1/8	1/16	1/32	1/32

$$\begin{aligned}\langle \text{Word length} \rangle &= 1/2 + (2 \times 1/4) + (3 \times 1/8) + (4 \times 1/16) + (2 \times 5 \times 1/32) \\ &= 1.9375\end{aligned}$$

$$\text{Shannon entropy of distribution} = - \sum_i p_i \log p_i = 1.9375$$

This compression method (“Huffman coding”) is optimal:

$$\langle \text{Word length} \rangle = \text{Shannon entropy}$$

Entropy of an **unknown** distribution?

Consider a bit-string where the letters are being drawn from an *unknown* stationary ergodic source:

AABABBBABABABBBABBABB

Lempel-Ziv compression (1977): make a **dictionary** out of this bit string:

A|AB|ABB|B|ABA|ABAB|BB|ABBA|BB

1	2	3	4	5	6	7	8	9
<i>A</i>	<i>AB</i>	<i>ABB</i>	<i>B</i>	<i>ABA</i>	<i>ABAB</i>	<i>BB</i>	<i>ABBA</i>	<i>BB</i>
$\emptyset A$	$1B$	$2B$	$\emptyset B$	$2A$	$5B$	$4B$	$3A$	7

← [borrowed from
Peter Shor's lecture notes]

0, 0|1, 1|10, 1|00, 1|010, 0|101, 1|100, 1|011, 0|0111

[Lempel, Ziv 1977, Wyner, Ziv 1993]: This compression scheme is also optimal. Unlike Huffman coding, requires just a long enough bit-string and nothing else.

Compression ratio = $\frac{\text{Length of compressed bit string}}{\text{Length of original bit string}}$ = Shannon entropy density

“Computable Information Density (CID)”

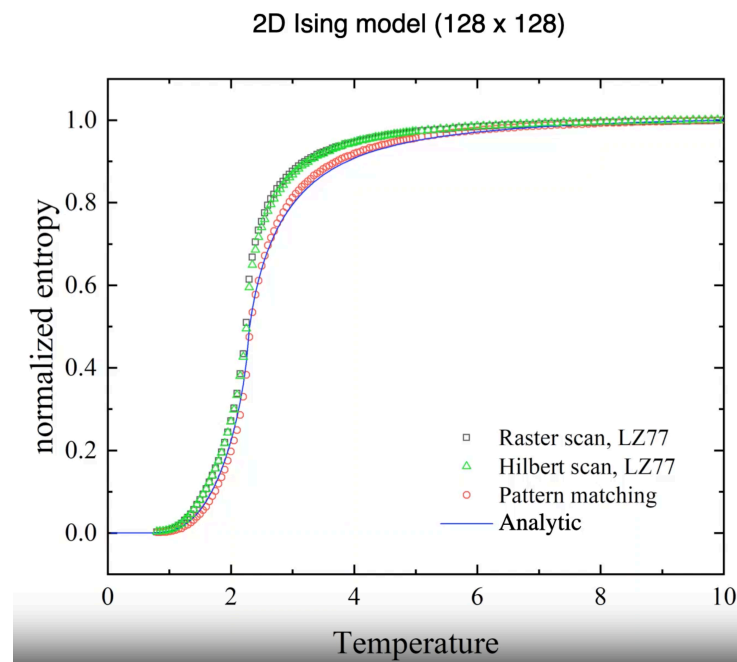
Physics by Zipping

PHYSICAL REVIEW X **9**, 011031 (2019)

Quantifying Hidden Order out of Equilibrium

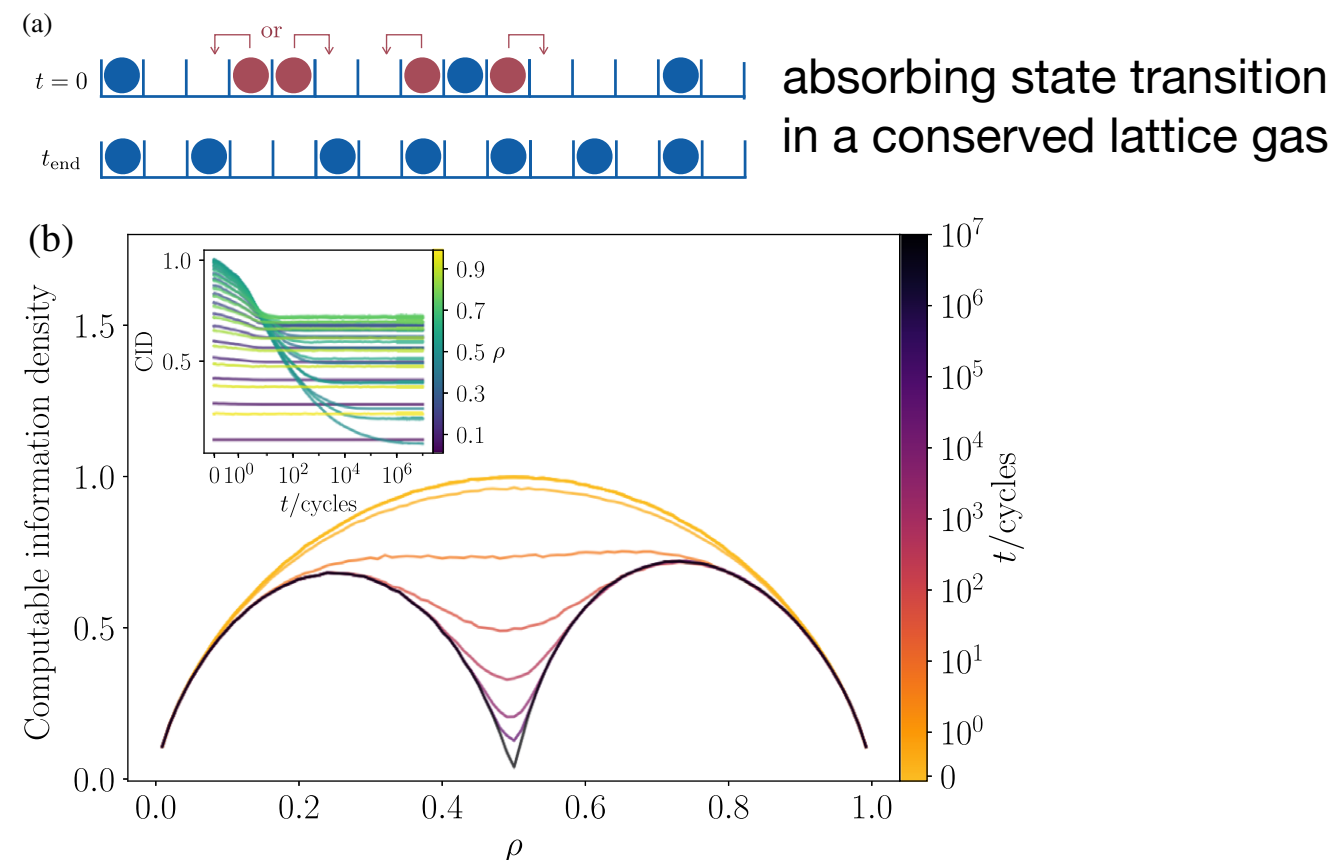
Stefano Martiniani,^{1,*} Paul M. Chaikin,^{1,†} and Dov Levine^{2,‡}

Equilibrium Entropy using compression



(from Dov Levine's seminar, 2022)

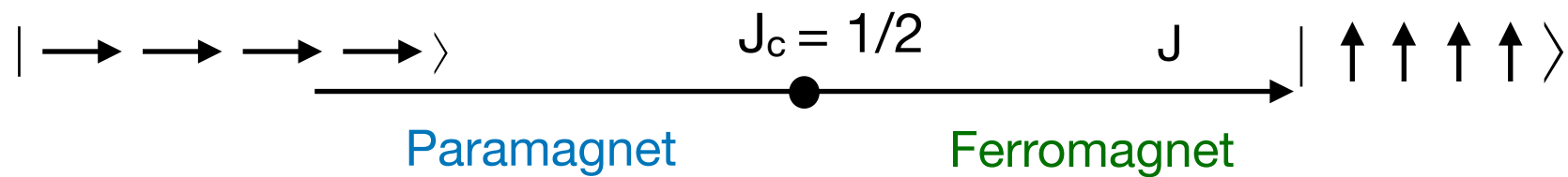
Non-equilibrium Entropy using compression



Zippering ground states

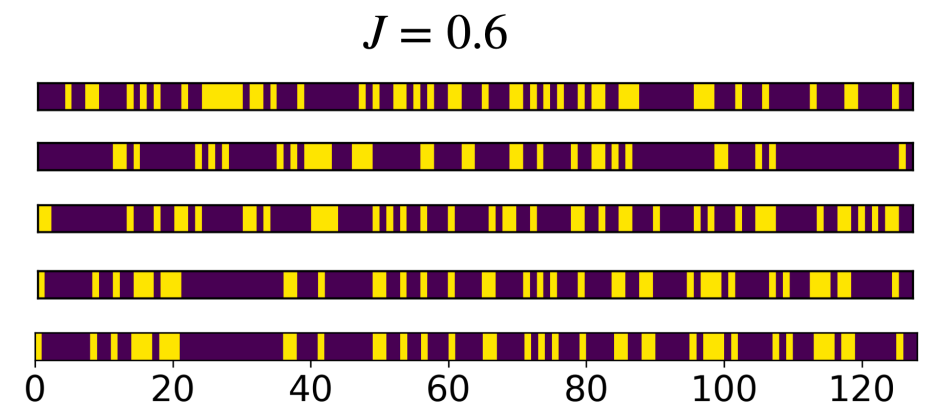
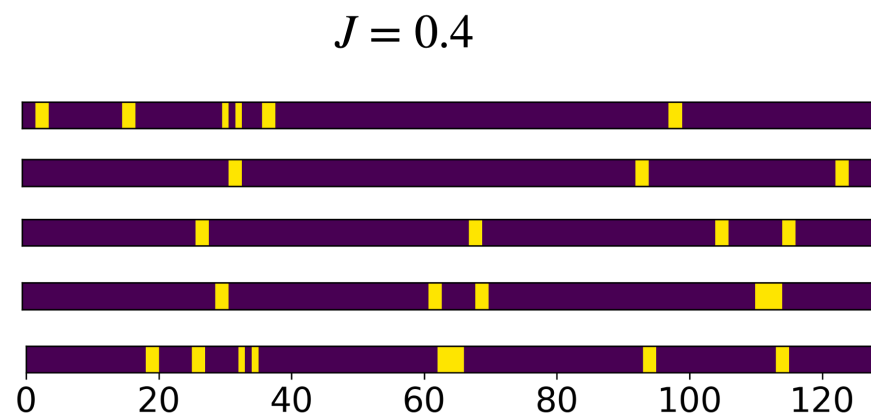
Zippping 1+1-D Quantum Ising Model

$$H = -(1 - J) \sum_i X_i - J \sum_{\langle i,j \rangle} Z_i Z_j$$



Let's take images in the X basis:

[Yu-Hsueh, TG 2025]



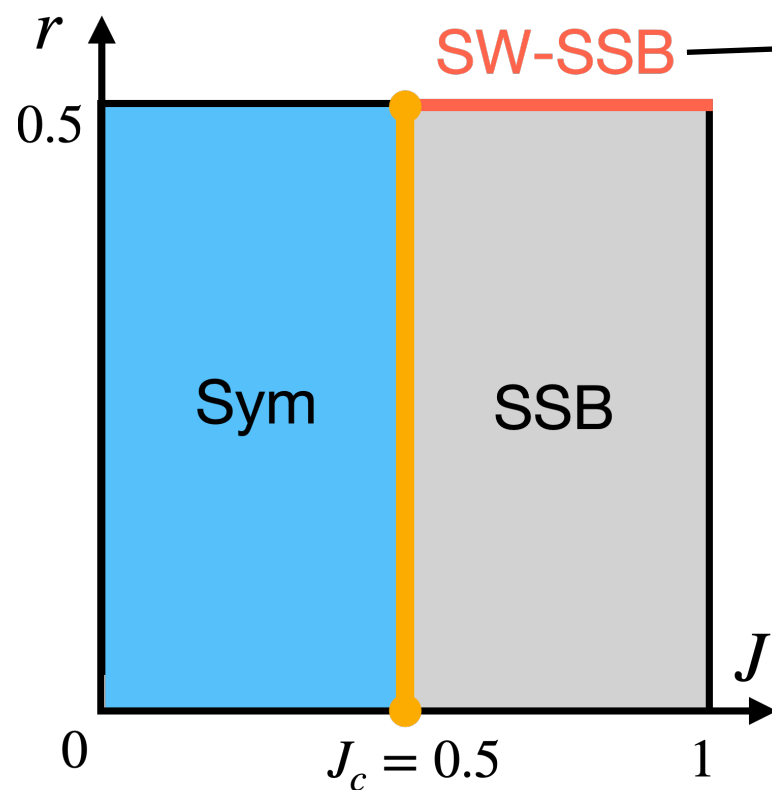
Denoting an image as $|x_j\rangle$, our goal is to estimate the diagonal entropy S_d :

$$S_d = - \sum_{x_j} \langle x_j | \rho | x_j \rangle \log(\langle x_j | \rho | x_j \rangle) = - \sum_{x_j} \rho_{x_j} \log \rho_{x_j}$$

Diagonal channel as decoherence

Consider the channel $\mathcal{E}_r[\cdot] = \prod_j \mathcal{E}_{j,r}[\cdot]$, $\mathcal{E}_{j,r}[\cdot] = (1-r)[\cdot] + rX_j[\cdot]X_j$

$\rho_d = \sum_{x_j} \rho_{x_j} |x_j\rangle\langle x_j|$ is obtained when $r = 1/2$



SW-SSB

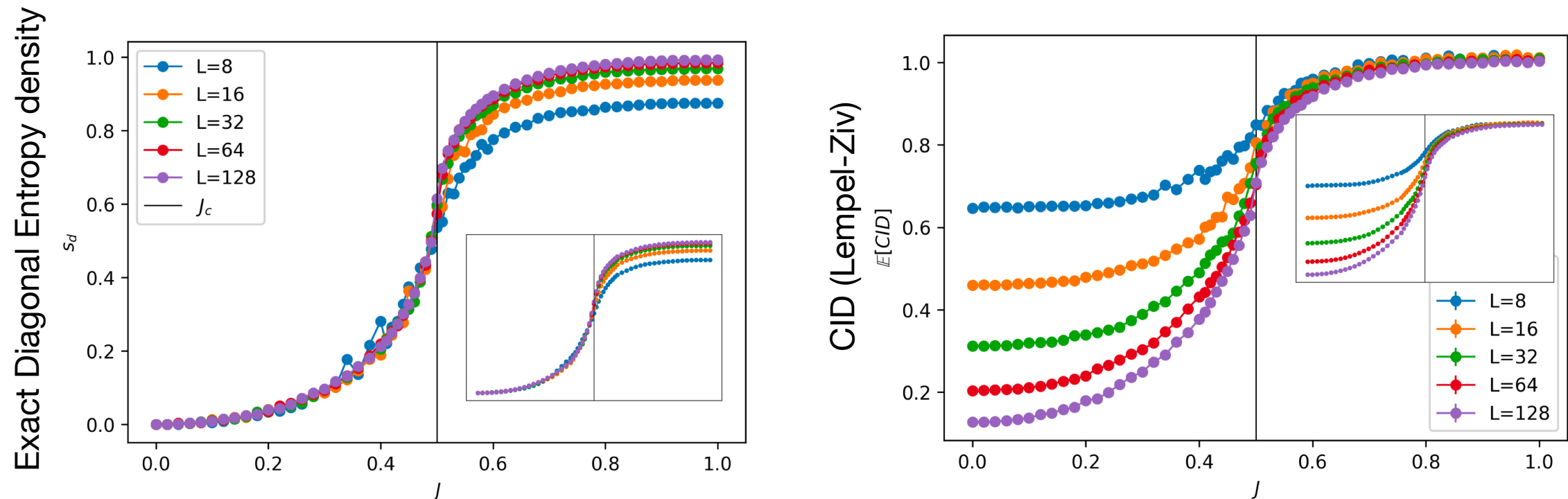
“Strong-to-weak SSB phase”

[Lee, Jian, Xu 2023;
Ma, Zhang, Bi, Cheng, Wang 2023]

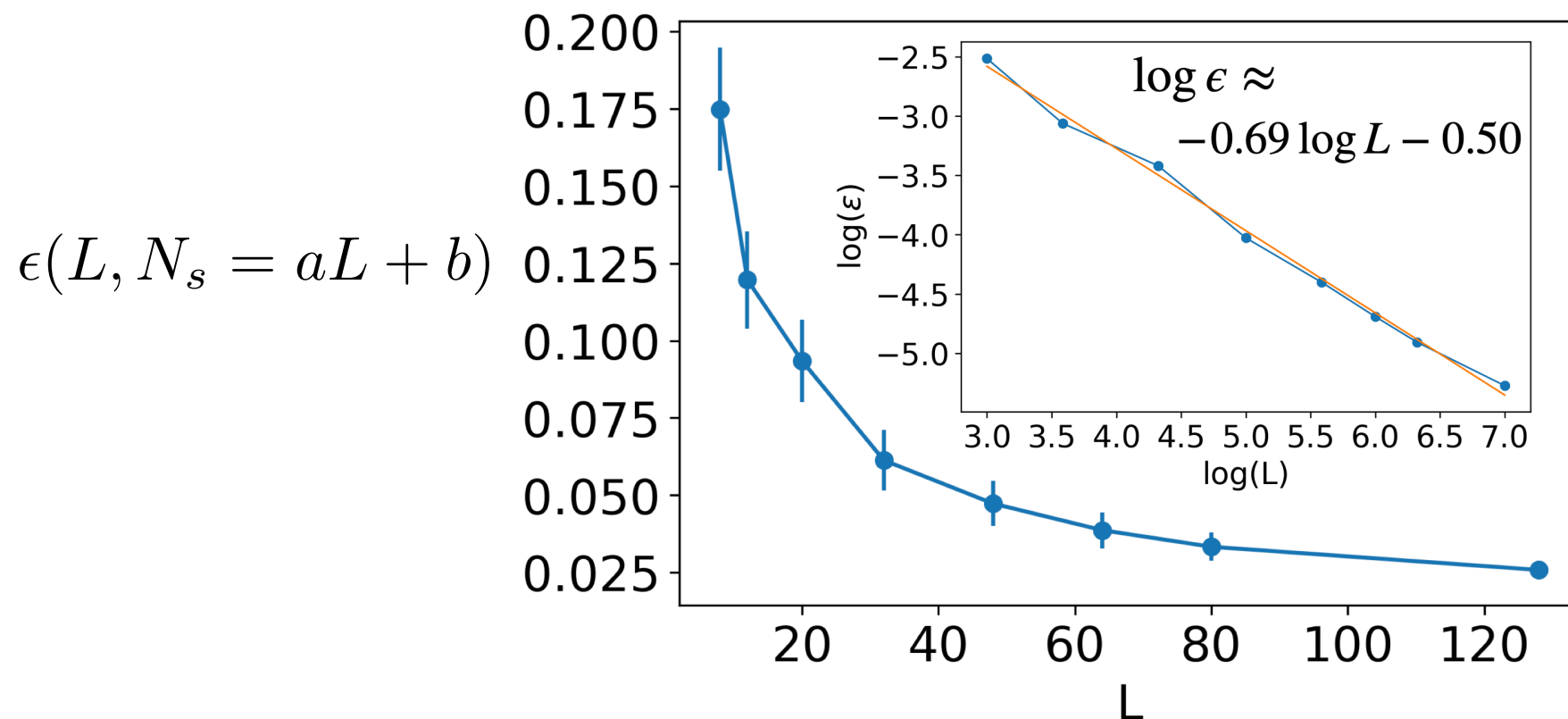
$$\lim_{|i-j| \rightarrow \infty} \text{tr}(\rho Z_i Z_j) = 0$$

$$\lim_{|i-j| \rightarrow \infty} F(\rho_d, Z_i Z_j \rho_d Z_i Z_j) = c > 0$$

Diagonal entropy vs Lempel-Ziv



How does $\epsilon(L, N_s) = |\mathbb{E}[CID](L, N_s) - s_d(L)|$ scale with L and N_s ?

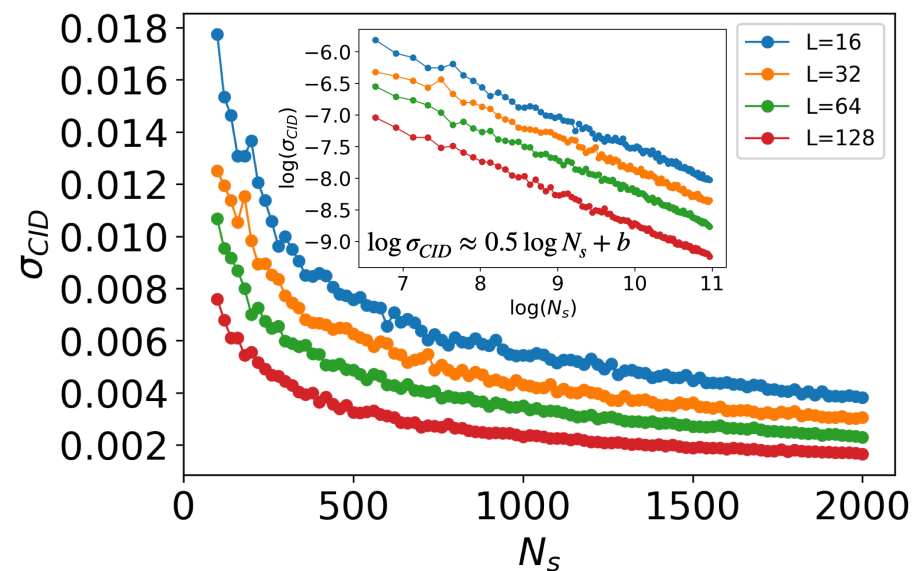


Sample complexity

Two sources of deviation between s_d and CID: finite size effect, and finite number of samples

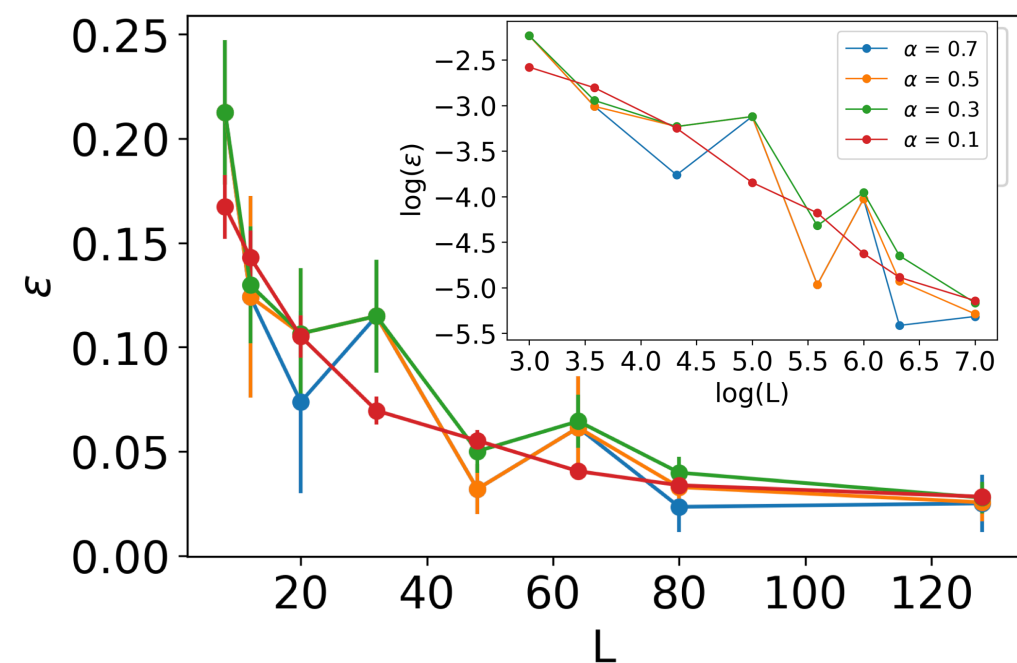
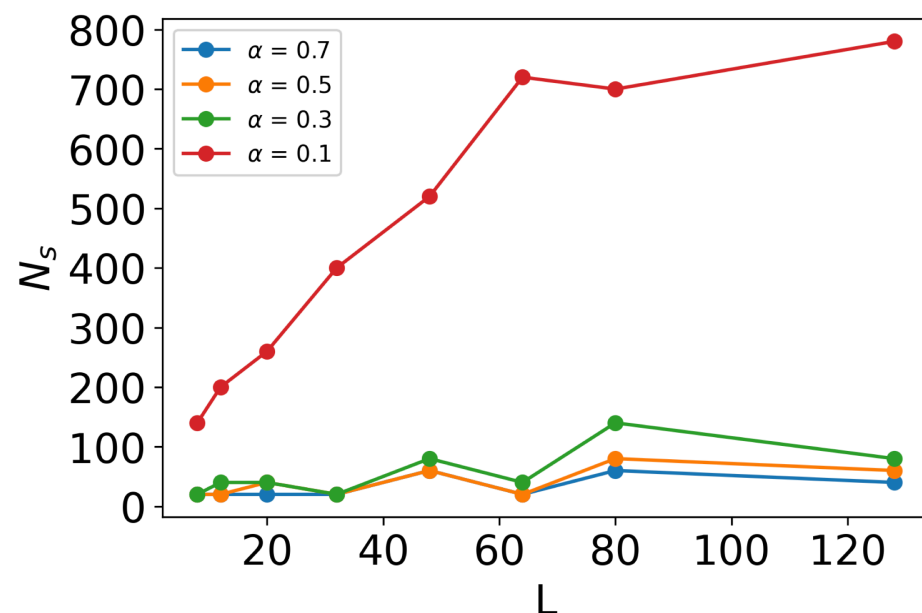
Standard deviation due to finite sampling

$$\sigma_{\text{CID}}(L, N_s) \approx \sigma_0(L) / \sqrt{N_s},$$



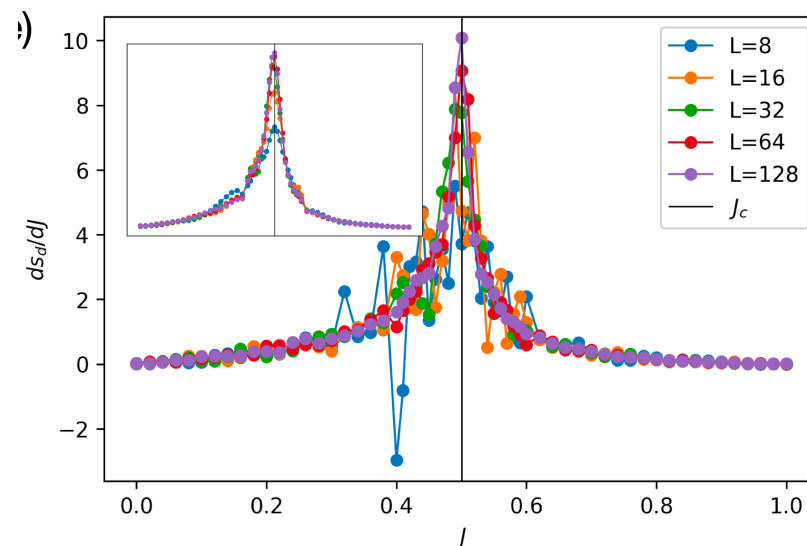
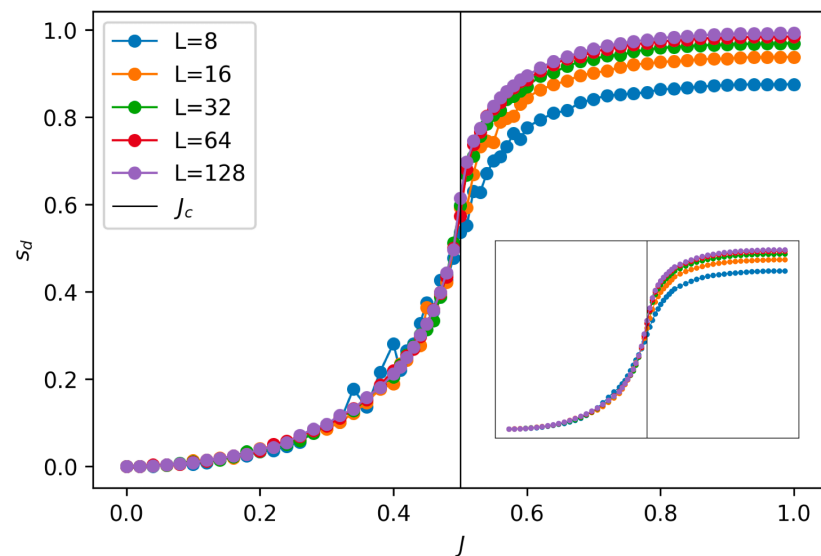
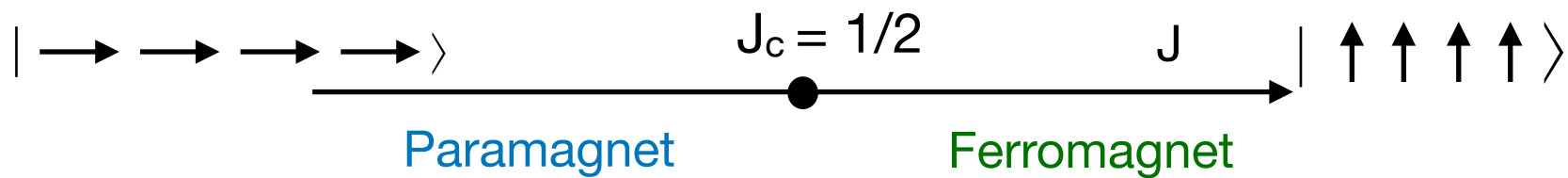
How many samples needed so that $\sigma_{\text{CID}}(L, N_s) \leq \alpha \epsilon(L, N_s)$

where $\epsilon(L, N_s) = |\mathbb{E}[\text{CID}](L, N_s) - s_d(L)|$



Locating phase transition

$$H = -(1 - J) \sum_i X_i - J \sum_{\langle i,j \rangle} Z_i Z_j$$

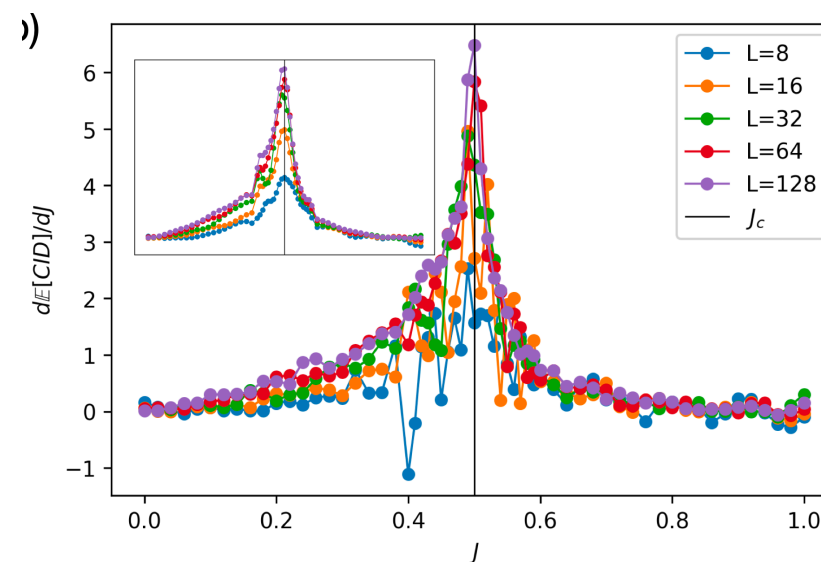
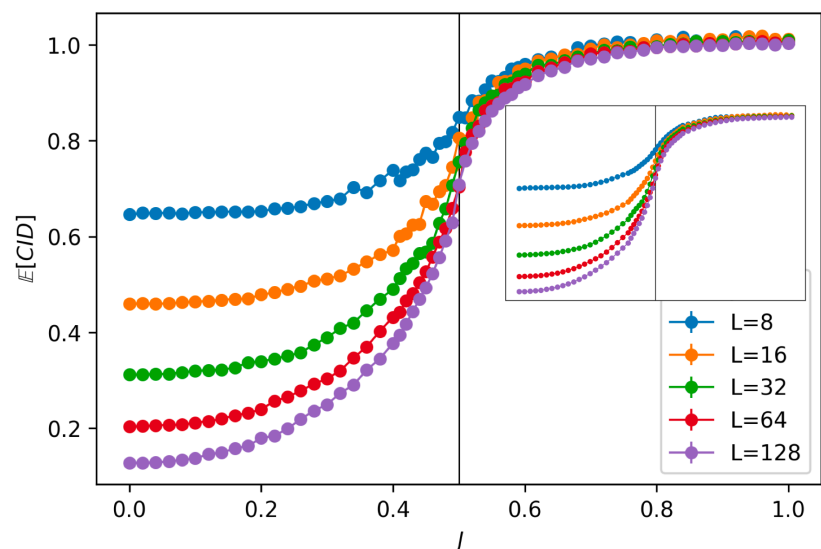


Field theory (replica calculation) predicts

$$S_d \sim |t| \log(1/|t|)$$

$$t = 1 - J/J_c$$

(diagonal entropy density can be mapped to the *boundary free energy* of critical 2d Ising model, calculated by [Ferdinand, Fisher, 1967])

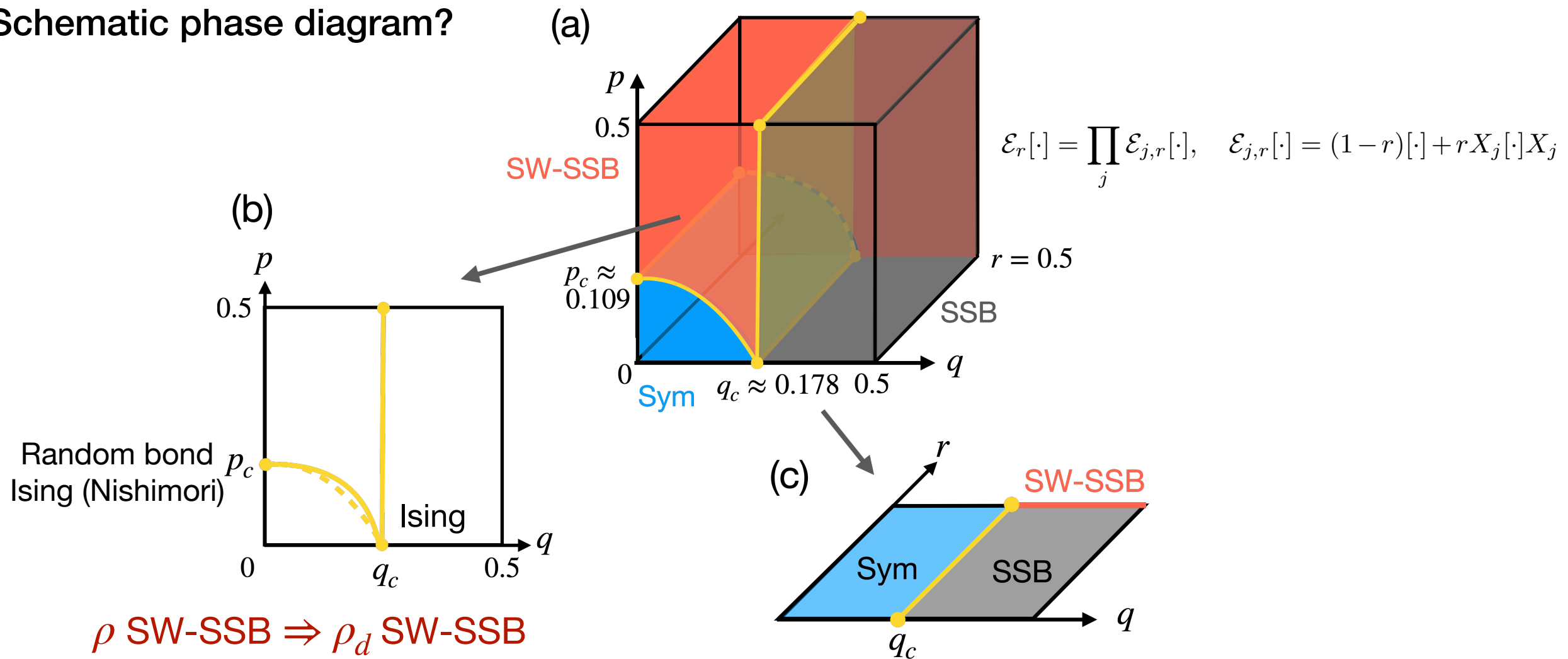


Zippering 2+1-D Decohered Ising State

We subject the pure state $|\Psi(q)\rangle = \prod_{\langle i,j \rangle} [(1-q)I + qZ_i Z_j] |x_j = 1\rangle$ on square lattice
[Castelnovo, Chamon 2008]

to decoherence $\mathcal{E}_{\langle i,j \rangle}(p)[\cdot] = (1-p)(\cdot) + pZ_i Z_j(\cdot)Z_i Z_j$

Schematic phase diagram?



$$\rho \text{ SW-SSB} \Rightarrow \rho_d \text{ SW-SSB}$$

$$\begin{aligned} F(\rho, Z_i Z_j \rho Z_i Z_j) &\leq F(\mathcal{E}_d[\rho], \mathcal{E}_d[Z_i Z_j \rho Z_i Z_j]) \\ &= F(\rho_d, Z_i Z_j \rho_d Z_i Z_j) \end{aligned}$$

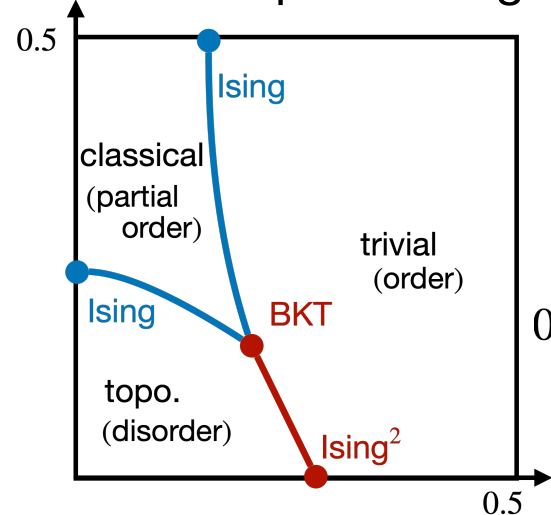
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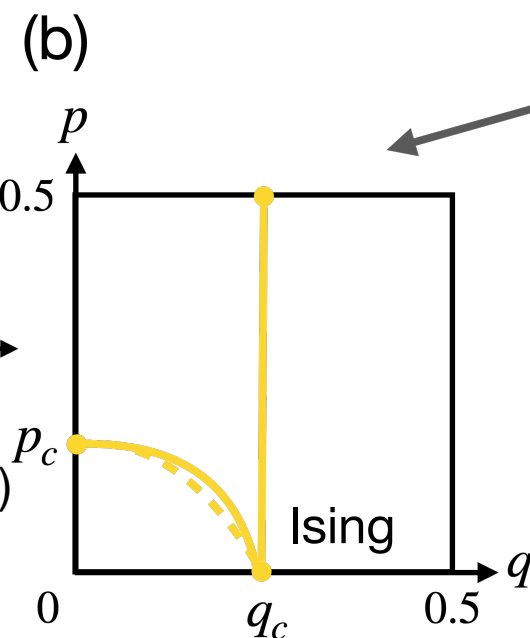
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Schematic phase diagram?

double state phase diagram (Chen, TG 2024)



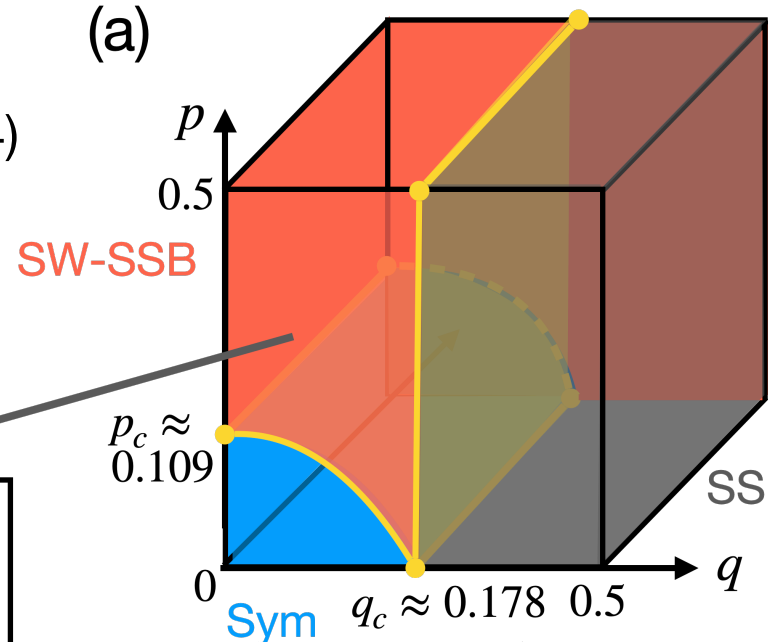
Random bond
Ising (Nishimori)



ρ SW-SSB $\Rightarrow \rho_d$ SW-SSB

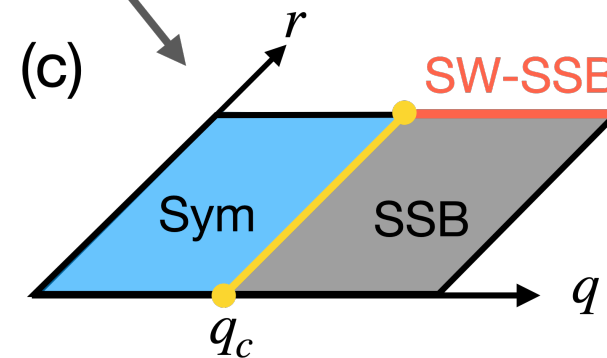
$$\begin{aligned} F(\rho, Z_iZ_j\rho Z_iZ_j) &\leq F(\mathcal{E}_d[\rho], \mathcal{E}_d[Z_iZ_j\rho Z_iZ_j]) \\ &= F(\rho_d, Z_iZ_j\rho_d Z_iZ_j) \end{aligned}$$

(a)



$$\mathcal{E}_r[\cdot] = \prod_j \mathcal{E}_{j,r}[\cdot], \quad \mathcal{E}_{j,r}[\cdot] = (1-r)[\cdot] + rX_j[\cdot]X_j$$

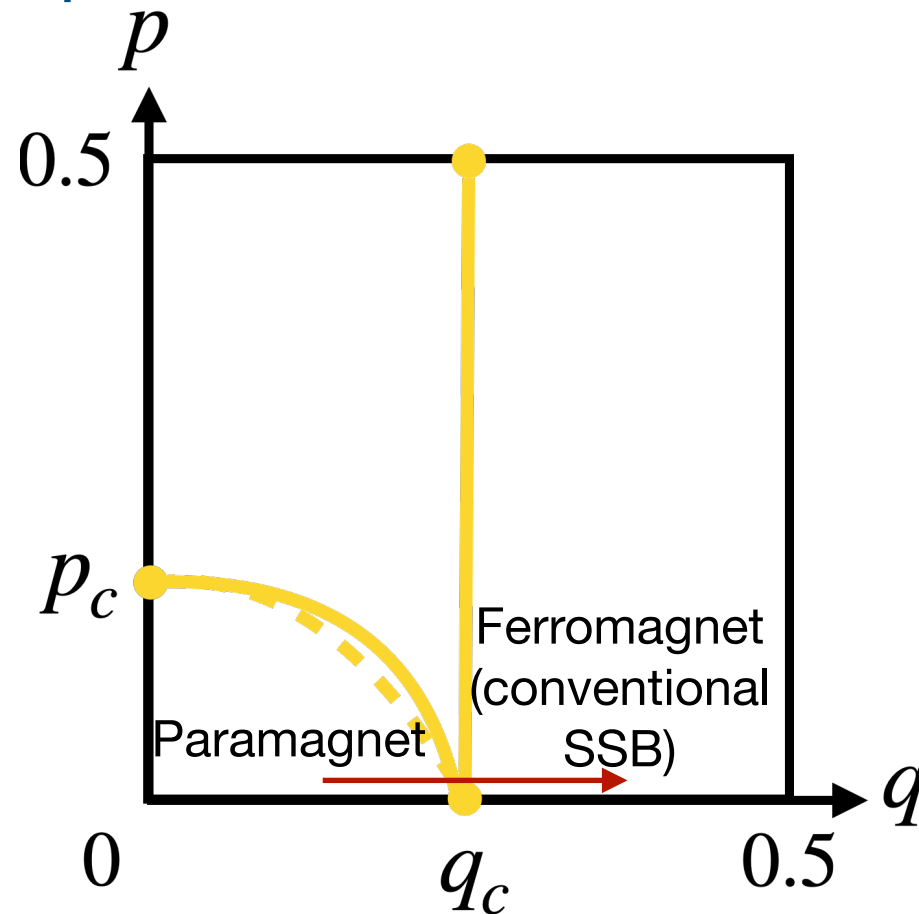
$r = 0.5$



Zippering 2+1-D Decohered Ising State

$$|\Psi(q)\rangle = \prod_{\langle i,j \rangle} [(1-q)I + qZ_i Z_j] |x_j = 1\rangle$$

First consider $p = 0$ line:



Field theory calculation predicts
diagonal entropy $s_d \sim t^2 \log(1/|t|)$
($t = |1 - q/q_c|$)

Diagonal density matrix along $p = 0$

$$\rho_{x_j} \propto \left| \sum_{s_{\tilde{i}}} e^{\beta \sum_{\langle \tilde{i}, \tilde{j} \rangle} J_{\langle \tilde{i}, \tilde{j} \rangle} s_{\tilde{i}} s_{\tilde{j}}} \right|^2 = \mathcal{Z}_{x_j}^2(q)$$

where $(\nabla \times J)_j = x_j$

$$\rho_d(q, 0) = \rho^2(0, p = q) / \text{tr}[\rho^2(0, p = q)]$$

$\rho^2(0, p)$ undergoes a separability transition
in the Ising universality class.

$\Rightarrow \rho_d$ undergoes SW-SSB transition along
the q axis in the Ising universality class

Brief outline of field-theory calculation for s_d

$$S_d = \lim_{n \rightarrow 1} \frac{\log \left(\sum_{x_j} \rho_{x_j}^n \right)}{1 - n} = \lim_{n \rightarrow 1} \frac{F(2n) - nF(2)}{1 - n}$$

$F(2n) = \log \left(\sum_{x_j} Z_{x_j}^{2n} \right)$ = Free energy of $2n-1$ copies of the Ising model interacting via

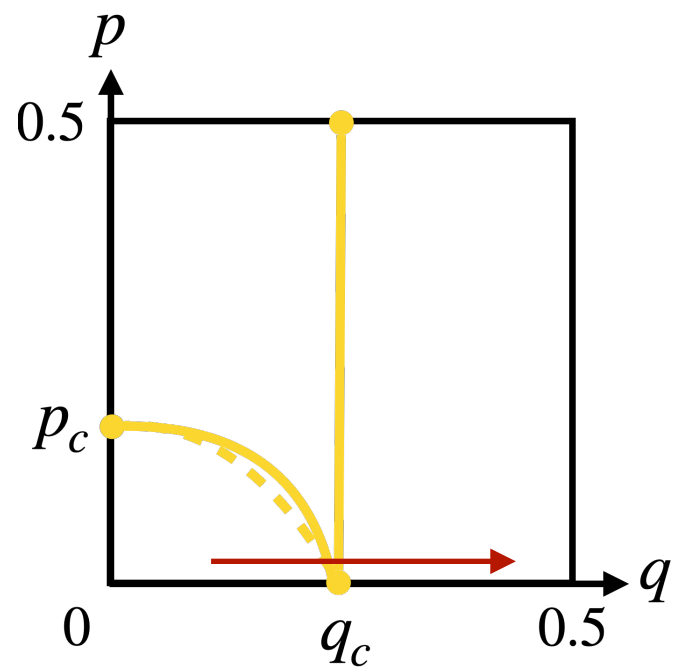
$$H_{\text{int}} = -K \sum_{\langle i, j \rangle} \left(\prod_{\alpha=1}^{2n-1} s_i^{(\alpha)} s_j^{(\alpha)} \right)$$

$$\mathcal{S}_{\text{int}} = \int d^D r \left(\lambda_0 \sum_{\alpha} \epsilon^{(\alpha)}(r) + \lambda \sum_{\alpha \neq \beta} \epsilon^{(\alpha)}(r) \epsilon^{(\beta)}(r) \right) \quad \text{(calculation mapped to Cardy's book's Chapter 8.3 on random bond Ising model)}$$

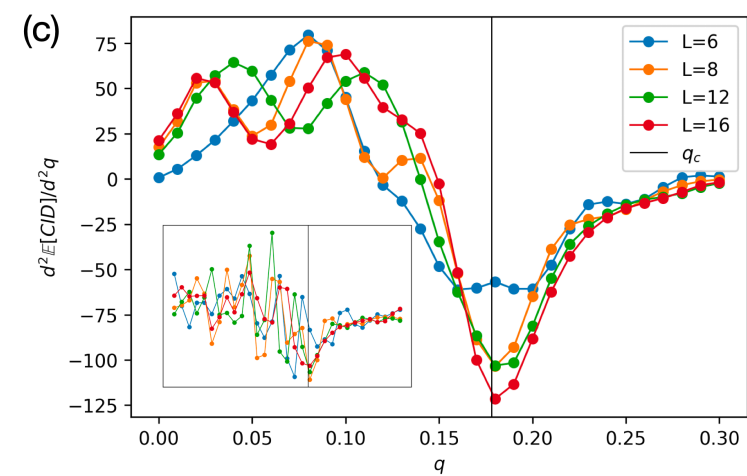
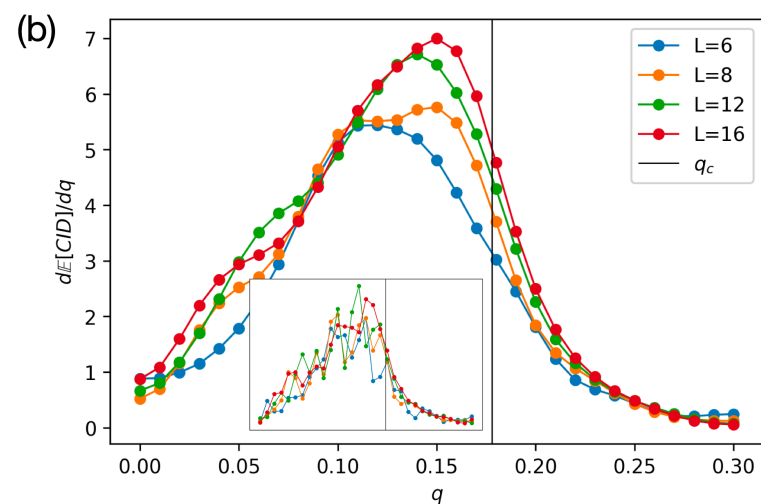
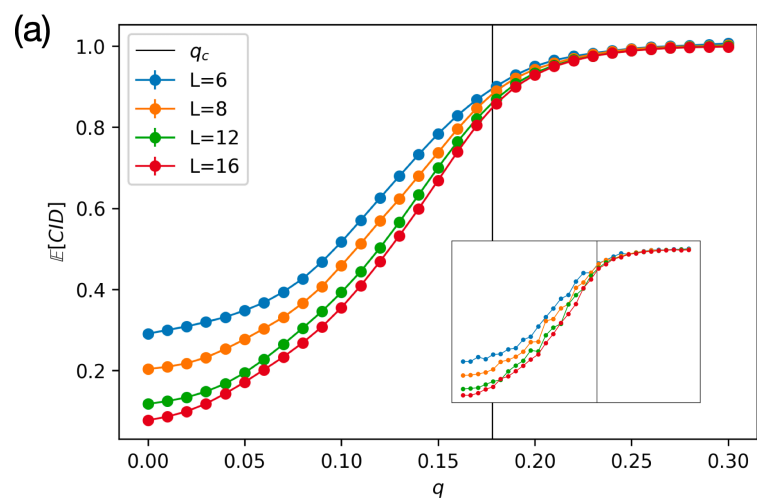
$$\frac{d\lambda}{dl} = \left(\frac{2}{\nu} - D \right) \lambda + (4(2n - 3) + 2C^2) \lambda^2 + O(\lambda^3) \quad \epsilon^{(\alpha)} \epsilon^{(\beta)} \sim \delta_{\alpha, \beta} + C \delta_{\alpha, \beta} \epsilon^{(\alpha)}$$

$\Rightarrow \lambda$ marginally irrelevant for $D = 2$ Ising model since $C = 0$ (Ising self-duality) and $\nu = 1$.

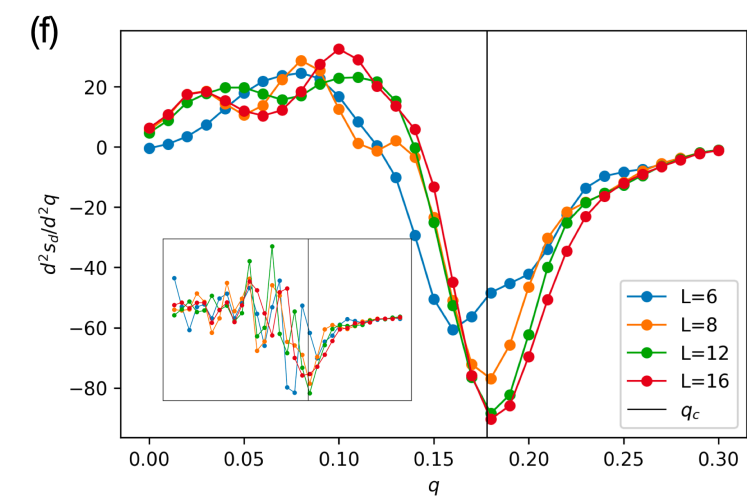
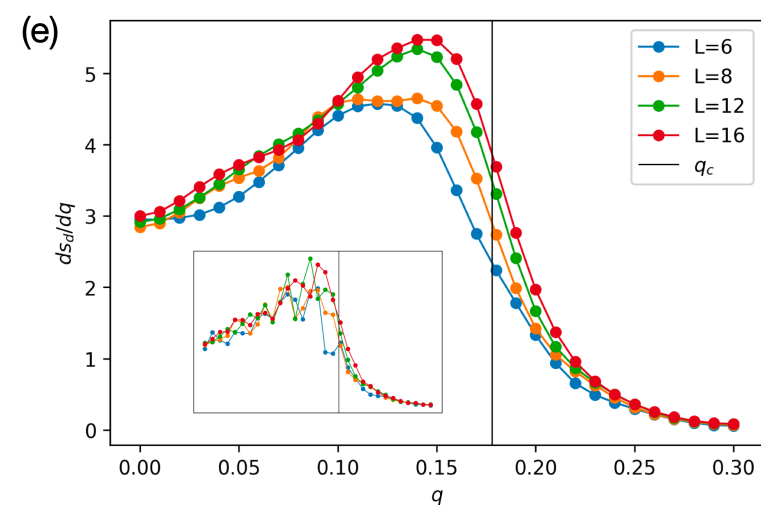
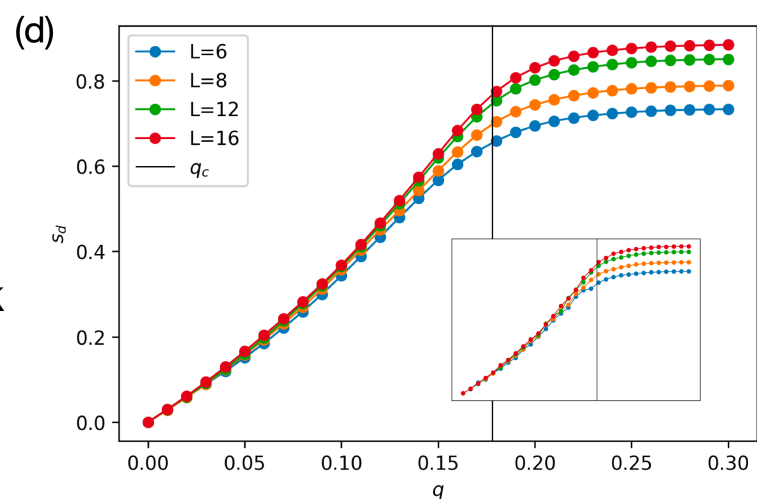
$\Rightarrow$ the leading singularity is the same as in the 2D Ising free energy, i.e., $t^2 \log(|t|)$.



Lempel-Ziv
→



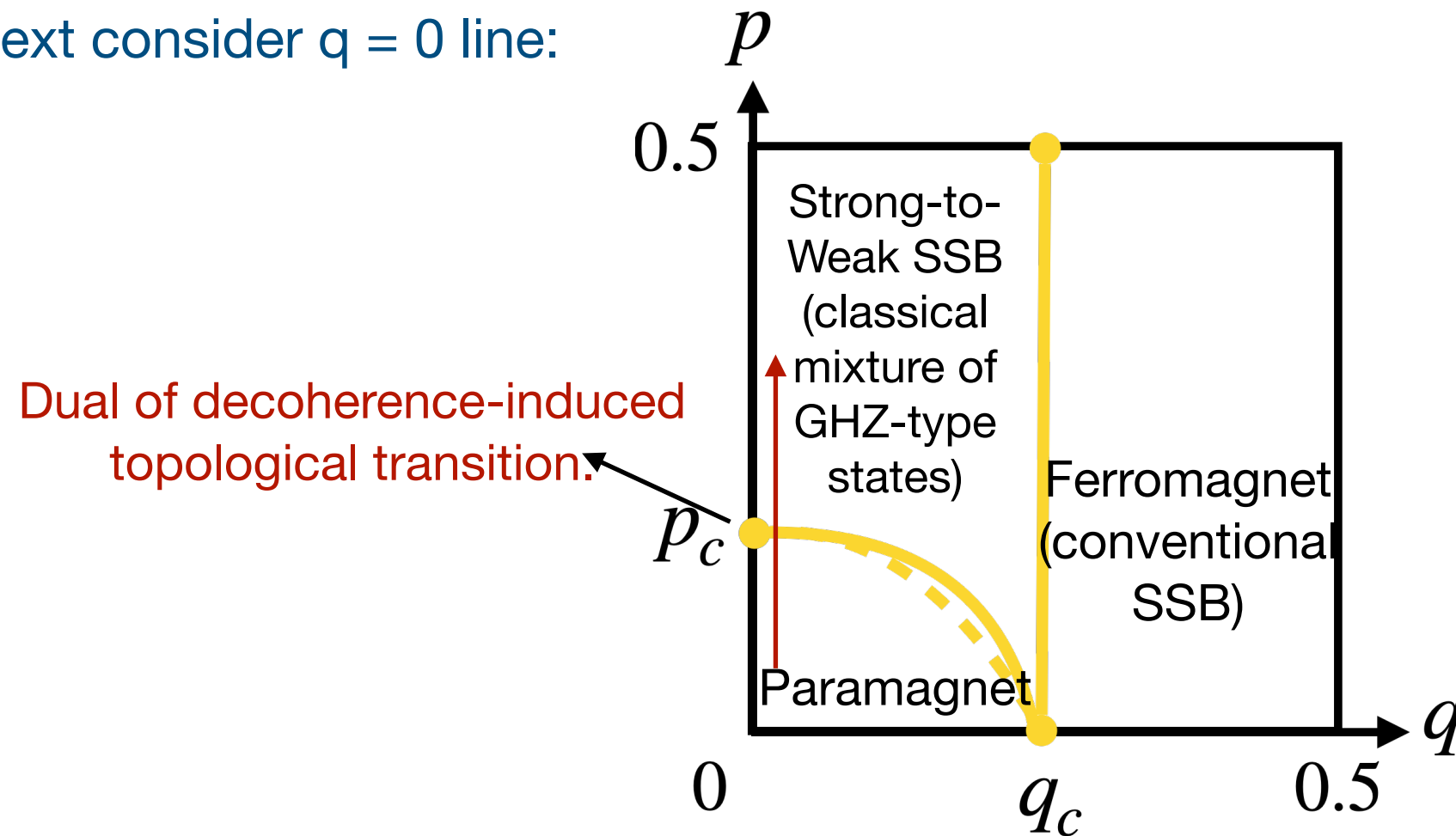
Exact
→
(tensor-network
contraction)



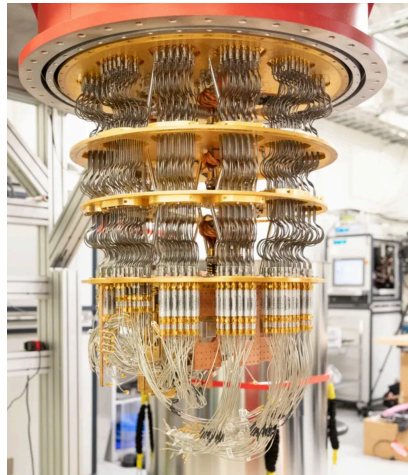
$$|\Psi(q)\rangle = \prod_{\langle i,j \rangle} [(1-q)I + qZ_i Z_j] |x_{\mathbf{j}} = 1\rangle \quad \text{on square lattice}$$

subjected to decoherence $\mathcal{E}_{\langle i,j \rangle}(p)[\cdot] = (1-p)(\cdot) + pZ_i Z_j(\cdot)Z_i Z_j$

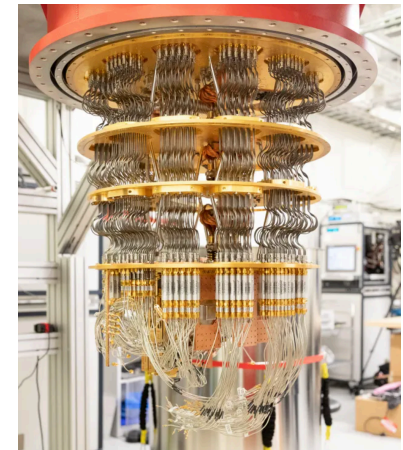
Next consider $q = 0$ line:



Brief detour on decoherence-induced topological transitions



$$= U^\dagger$$

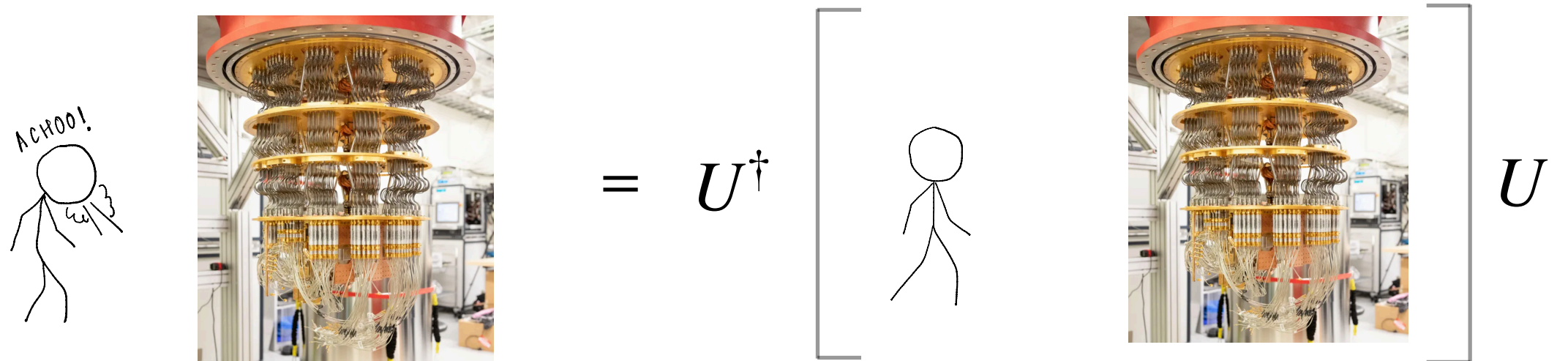


$$U$$

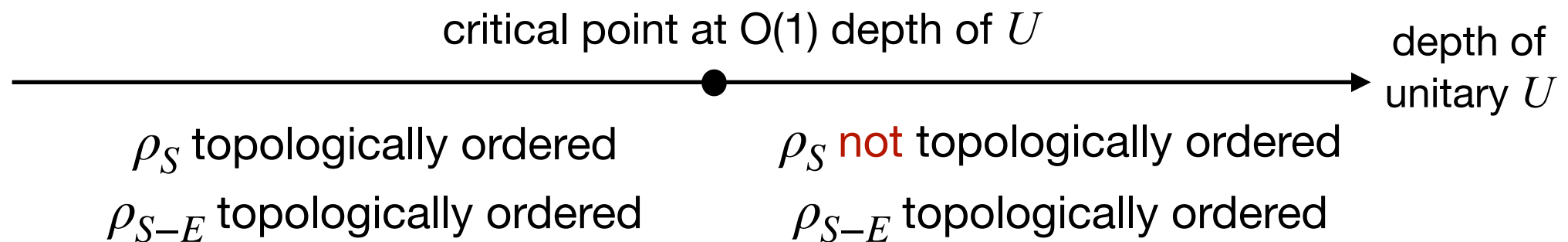
$$\rho_S = \text{tr}_E \rho_{S-E}$$

S = quantum computer,
 E = experimentalist

Brief detour on decoherence-induced topological transitions



$$\rho_S = \text{tr}_E \rho_{S-E} \quad \begin{array}{l} S = \text{quantum computer,} \\ E = \text{experimentalist} \end{array}$$



Correlation functions of local operators completely analytic across the transition!
Requires observables non-linear in the density matrix.

[Dennis, Kitaev, Landahl, Preskill 2001; Fan, Bao, Altman, Vishwanath 2023; Lee, Jian, Xu 2023,...]

Entanglement across decoherence induced topological transitions

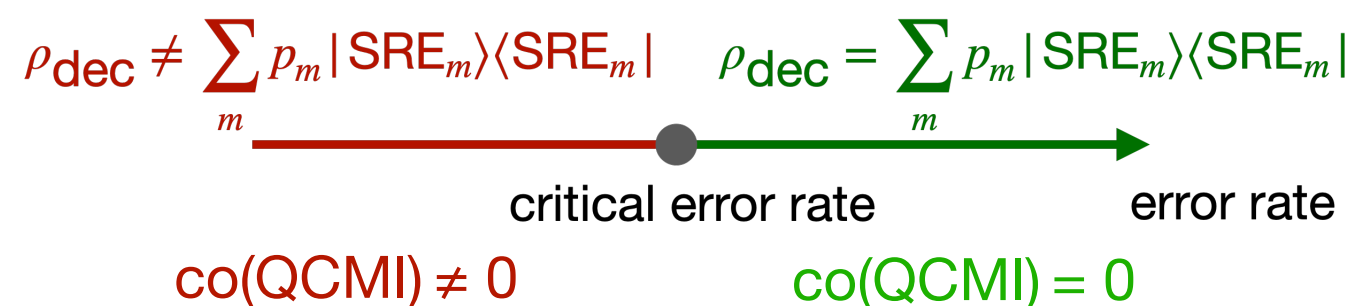
For a mixed-state ρ , define $\text{co}(\text{QCMI}) = \min(\sum_i p_i \gamma(|\psi_i\rangle))$

where the minimum is taken over *all* possible decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

and $\gamma(|\psi_i\rangle)$ is the topological entanglement entropy for the state $|\psi_i\rangle$

$\text{co}(\text{QCMI})$ = “long-range part of mixed-state entanglement”.

Equals zero if and only if ρ admits decomposition in terms of pure states with zero TEE.

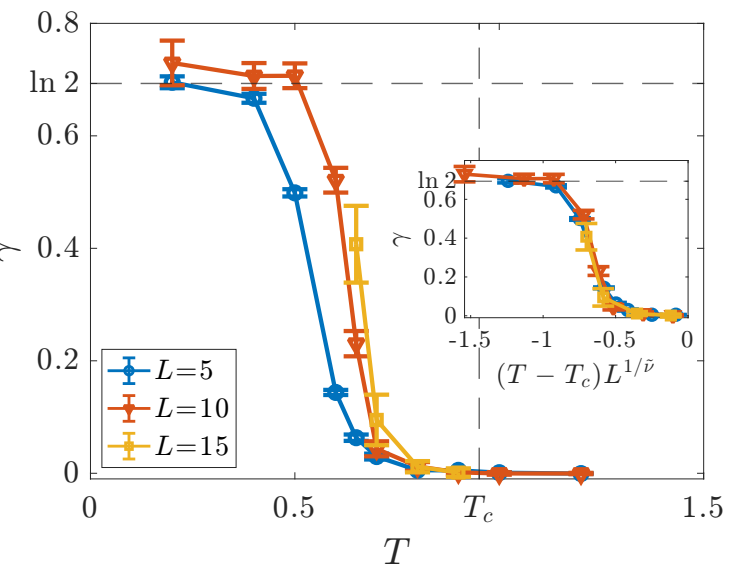


[Yu-Hsueh, TG, 2023

Wang, Song, Meng, TG, 2024]

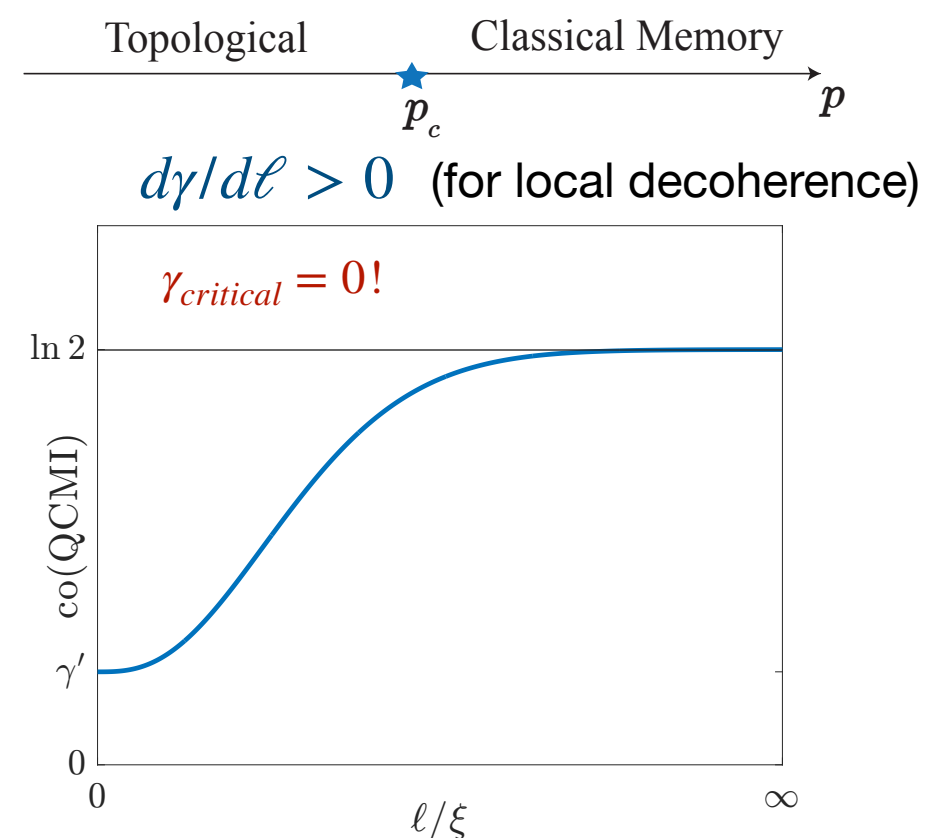
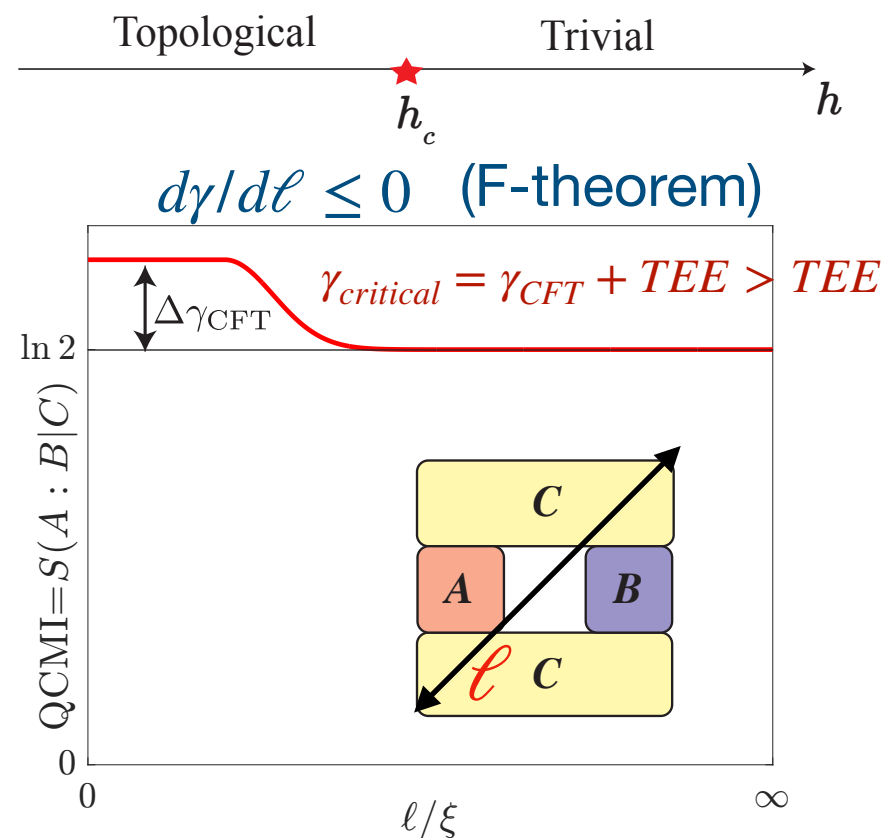
Decoherence induced transitions Vs Ground-state transitions

$$\text{co}(\text{QCMI})[\rho_{ABCD}] = \inf\left\{\sum_i p_i \gamma(|\psi_i\rangle_{ABCD})\right\}$$



(a) $H(h) = H_{2d \text{ Toric}} + h \sum_e X_e$
 $\rho(h) = \text{ground state of } H(h)$

(b) $\mathcal{E}_e[\rho_0] = pZ_e\rho_0Z_e + (1-p)\rho_0$
 $\rho(p) = \prod_e \mathcal{E}_e[\rho(h=0)]$

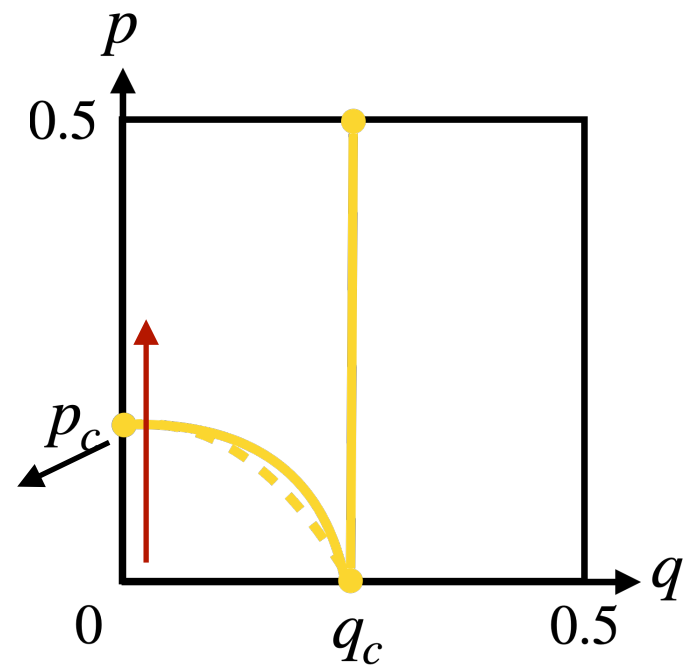


Back to zipping states...

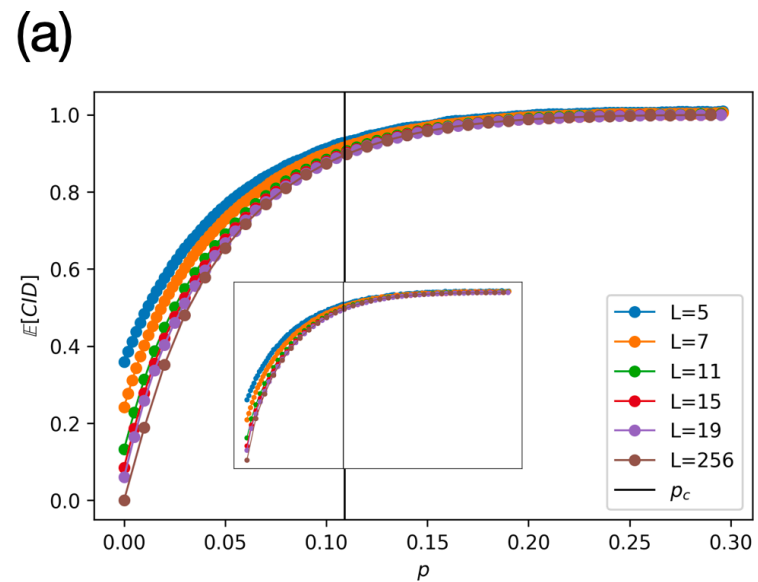
$$|\Psi(q)\rangle = \prod_{\langle i,j \rangle} [(1-q)I + qZ_i Z_j] |x_j = 1\rangle$$

$$\mathcal{E}_{\langle i,j \rangle}(p)[\cdot] = (1-p)(\cdot) + pZ_i Z_j(\cdot)Z_i Z_j$$

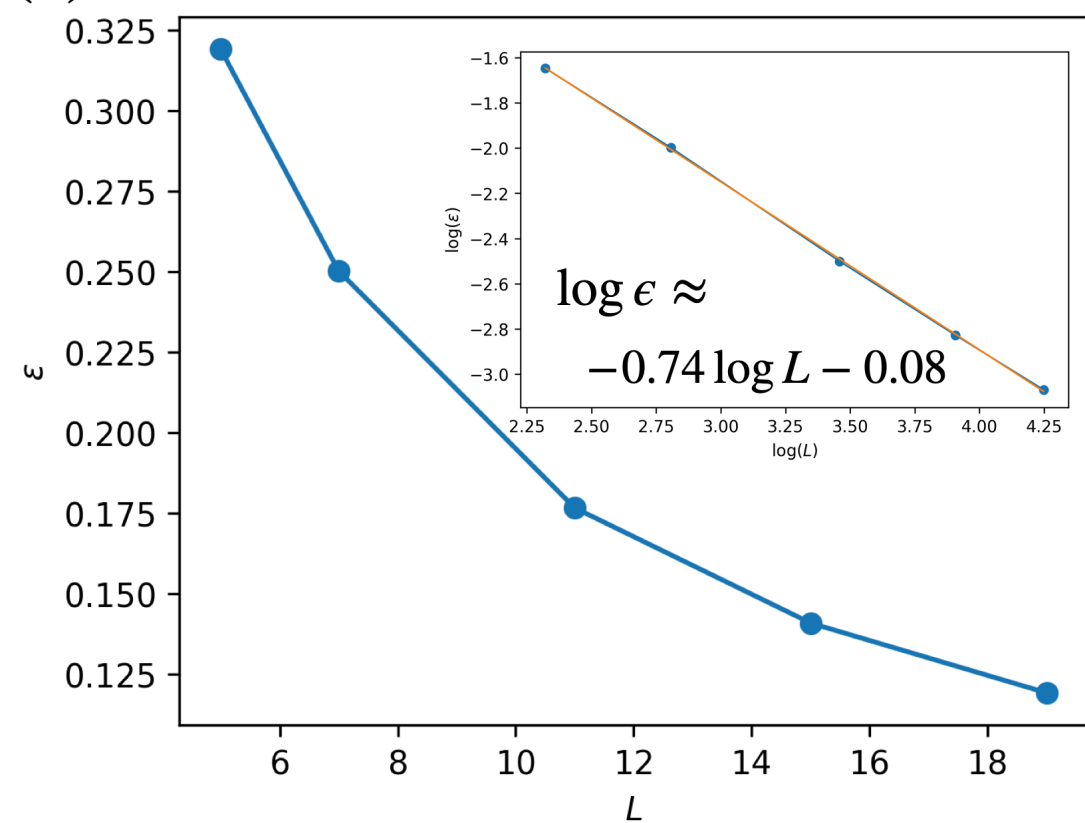
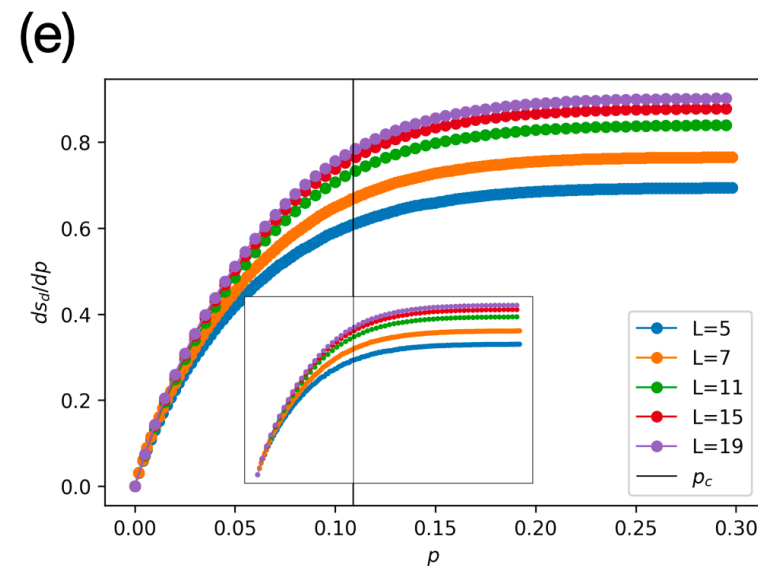
Dual of decoherence-induced
topological transition.



Lempel-Ziv
→



Exact
→



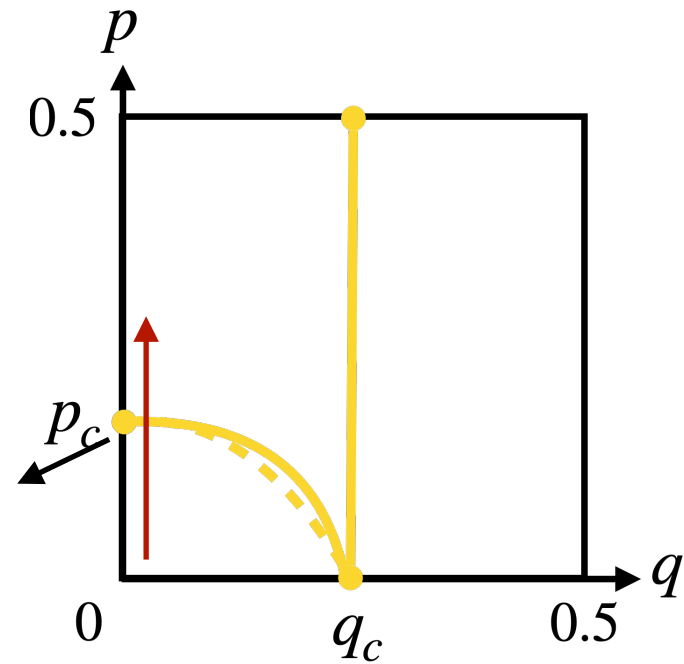
Lempel-Ziv works for the diagonal
entropy density, but not subleading terms.

Back to zipping states...

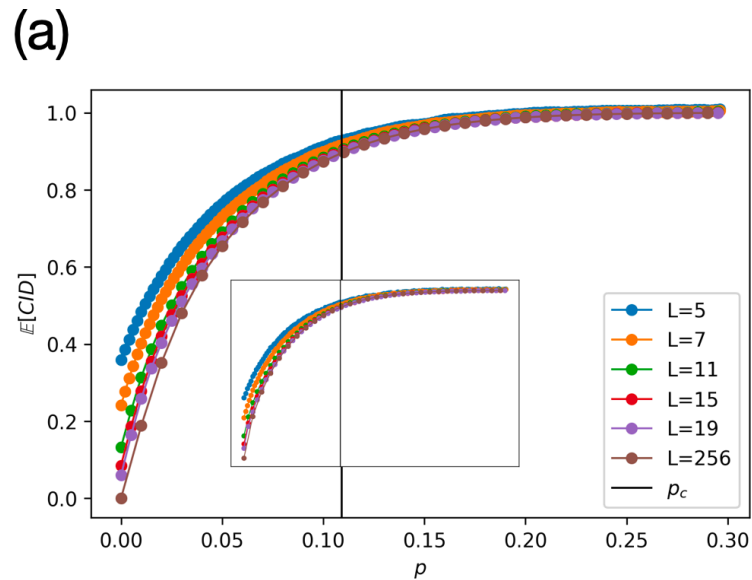
$$|\Psi(q)\rangle = \prod_{\langle i,j \rangle} [(1-q)I + qZ_i Z_j] |x_j = 1\rangle$$

$$\mathcal{E}_{\langle i,j \rangle}(p)[\cdot] = (1-p)(\cdot) + pZ_i Z_j(\cdot)Z_i Z_j$$

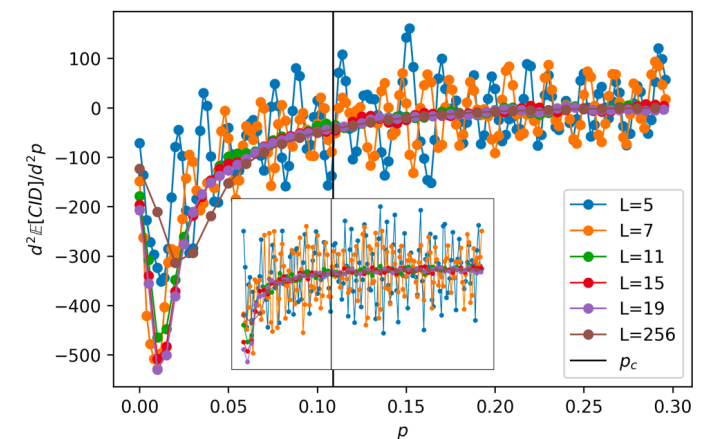
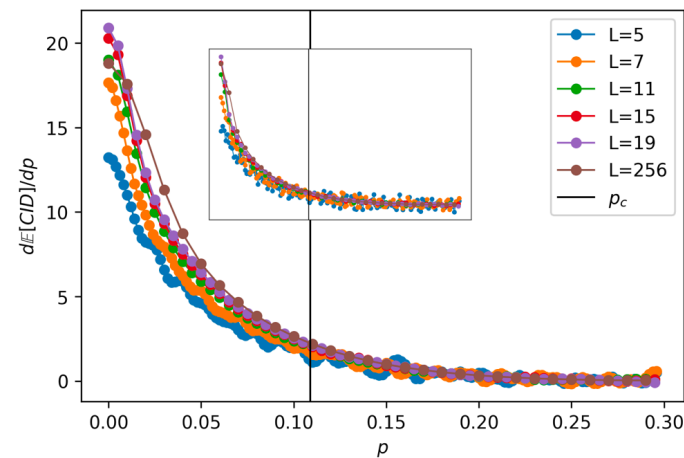
Dual of decoherence-induced
topological transition.



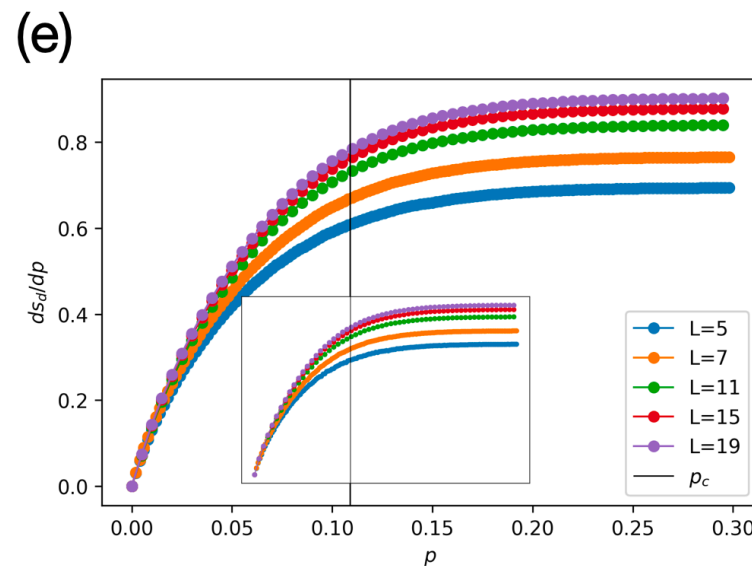
Lempel-Ziv
→



Singularity expected in the third-derivative of s_d .
Not enough resolution to resolve it :(



Exact
→

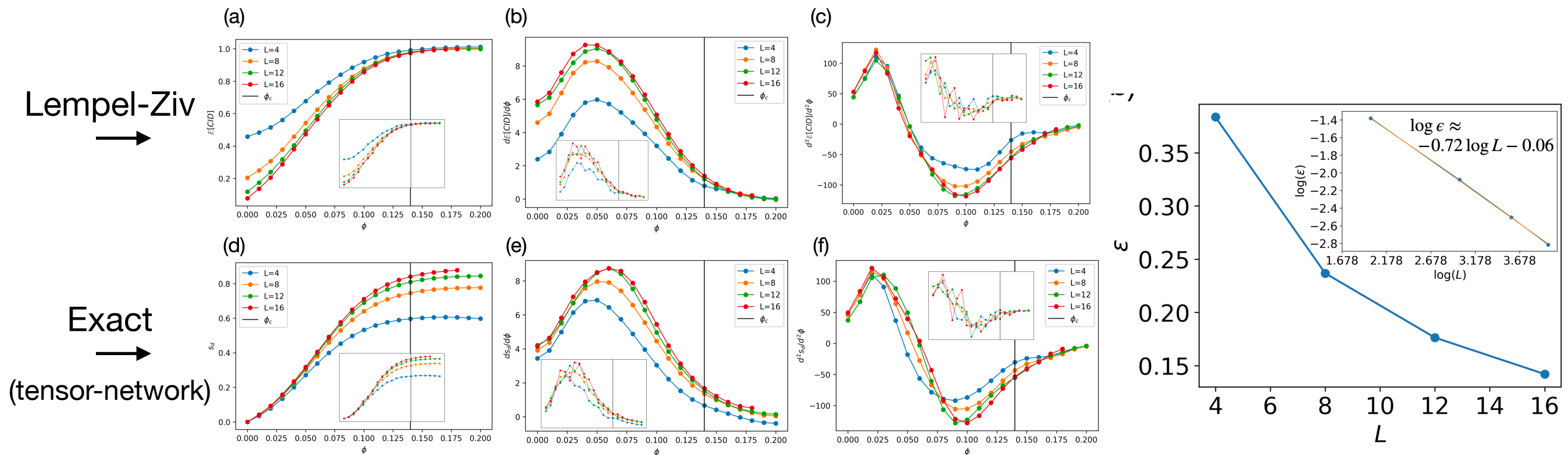


Further, Lempel-Ziv works for the diagonal
entropy density, but not subleading terms :(

Another example: coherent errors

Subject $|x_j = 1\rangle\langle x_j = 1|$ to the following Ising-symmetry preserving unitary

$$U(\phi) = e^{i\phi \sum_{\langle i,j \rangle} Z_i Z_j} \text{ and then measure in the X-basis}$$



Exploration limited by ability to generate images with Born rule:
good opportunity for experiments!

An explicit example where the method fails

Consider $\rho = \frac{1}{2} (|x_A\rangle\langle x_A| + |x_B\rangle\langle x_B|)$

$|x_A\rangle, |x_B\rangle$ are two distinct product states generated as follows: with probability p , $x_A(i) = 1$, and with probability $1 - p$, $x_A(i) = -1$. Same procedure for $|x_B\rangle$.

Clearly, $S_d = \log(2) \Rightarrow s_d = \log(2)/N \approx 0$,
but Lempel-Ziv's $\text{CID} = -p \log(p) - (1 - p)\log(1 - p)$

Reason: lack of translational invariance (“stationarity”).

When does the method work?

[Wyner, Ziv 1993] showed that Lempel-Ziv works for **stationary, ergodic processes**. This seemingly implies that the method gives the correct diagonal entropy density if the state is translationally invariant, and bit-strings generated do not have non-trivial subsequences of non-zero probability that are translationally invariant.

Even non-Markov chains can be stationary and ergodic. Therefore, even if the conditional mutual information in the diagonal density matrix does not decay exponentially (while satisfying stationarity and ergodicity), then the method likely continues to work. Our numerics results are consistent with this expectation (e.g. the Nishimori critical point) but we are limited by size.

More work needed to analytically understand these conditions for physical states.

Summary and questions

We motivated image compression method to efficiently estimate diagonal entropy of a quantum state — it seems to track diagonal entropy density rather well in the physical examples we tried.

Image compression methods to estimate relative entropy between two distributions? (guess: use one image as a “dictionary” for another).

RG/field-theory understanding of Lempel-Ziv compression?

If there exists no other method to estimate diagonal entropy,
how would one know that it has been estimated correctly in an experiment?

Can one encode NP hard problems in the estimation of diagonal entropy?

Cold atoms are a natural platform for imaging, but can other methods also work, e.g., simultaneous NV-sensing of $N \gg 1$ spins in a chosen basis?

One may not need single-site resolution — a course grained image may also work to extract universal physics.

Scaling of $\epsilon = |s_d - CID|$ at
the 1+1-D TFIM critical point

